

Quadratic Conformal Superalgebras¹

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Abstract

In this paper, we shall classify “quadratic” conformal superalgebras by certain compatible pairs of a Lie superalgebra and a Novikov superalgebra. Four general constructions of such pairs are given. Moreover, we shall classify such pairs related to simple Novikov algebras.

1 Introduction

The notion of conformal superalgebra was formulated by Kac [Ka2], which is equivalent to the notion of linear Hamiltonian operator in Gel’fand-Dikii-Dorfman’s theory in [GD1-2] and [GDo1-3]. Conformal superalgebras play important roles in quantum field theory (e.g. cf. [Ka2]) and vertex operator superalgebras (e.g. cf. [Ka2], [X7]). In some sense, conformal superalgebras are generalizations of affine Kac-Moody algebras and the Virasoro algebra. In this paper, we shall study a special class of conformal superalgebras, which we call “quadratic conformal superalgebras.” Below, we give a more detailed introduction.

Throughout this paper, all the vector spaces are assumed over \mathbb{C} , the field of complex numbers. Denote by \mathbb{C}^+ the additive group of \mathbb{C} . For two vector spaces V_1 and V_2 , we denote by $LM(V_1, V_2)$ the space of linear maps from V_1 to V_2 . Moreover, we denote by \mathbb{Z} the ring of integers and by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2. When the context is clear, we use $\{0, 1\}$ to denote the elements of \mathbb{Z}_2 . We shall also use the following operator of taking residue:

$$\text{Res}_z(z^n) = \delta_{n,-1} \quad \text{for } n \in \mathbb{Z}. \quad (1.1)$$

Furthermore, all the binomials are assumed to be expanded in the nonnegative powers of the second variable.

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A *conformal superalgebra* $R = R_0 \oplus R_1$ is a \mathbb{Z}_2 -graded $\mathbb{C}[\partial]$ -module with a \mathbb{Z}_2 -graded linear map $Y^+(\cdot, z) : R \rightarrow LM(R, R[z^{-1}]z^{-1})$ satisfying:

$$Y^+(\partial u, z) = \frac{dY^+(u, z)}{dz} \quad \text{for } u \in R; \quad (1.2)$$

$$Y^+(u, z)v = (-1)^{ij} \text{Res}_x \frac{e^{x\partial} Y^+(v, -x)u}{z-x}, \quad (1.3)$$

$$Y^+(u, z_1)Y^+(v, z_2) - (-1)^{ij}Y^+(v, z_2)Y^+(u, z_1) = \text{Res}_x \frac{Y^+(Y^+(u, z_1-x)v, x)}{z_2-x} \quad (1.4)$$

for $u \in R_i$; $v \in R_j$. We denote by $(R, \partial, Y^+(\cdot, z))$ a conformal superalgebra. When $R_1 = \{0\}$, we simply call R a *conformal algebra*.

The above definition is the equivalent generating-function form to that given in [Ka2], where the author used the component formulae with $Y^+(u, z) = \sum_{n=0}^{\infty} u_{(n)}z^{-1}$.

Suppose that $(R, \partial, Y^+(\cdot, z))$ is a conformal superalgebra that is a free $\mathbb{C}[\partial]$ -module over a \mathbb{Z}_2 -graded subspace V , namely

$$R = \mathbb{C}[\partial]V \quad (\cong \mathbb{C}[\partial] \otimes_{\mathbb{C}} V). \quad (1.5)$$

Let m be a positive integer. The algebra R is called a *homogeneous conformal superalgebra of degree m* if for any $u, v \in V$,

$$Y^+(u, z)v = \sum_{j=1}^m \partial^{m-j} w_j z^{-j} \quad \text{with } w_j \in V. \quad (1.6)$$

A *Lie superalgebra* L is a \mathbb{Z}_2 -graded algebra $L = L_0 \oplus L_1$ with the operation $[\cdot, \cdot]$ satisfying

$$[u, v] = -(-1)^{ij}[v, u], \quad [w, [v, u]] = [[w, v], u] - (-1)^{ij}[[w, u], v] \quad (1.7)$$

for $u \in L_i$, $v \in L_j$ and $w \in L$. It is well known that a homogeneous conformal superalgebra of degree 1 is equivalent to a Lie superalgebra (e.g., cf. [Gdo1], [Ka2]).

A *Novikov superalgebra* is a \mathbb{Z}_2 -graded vector space $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with an operation “ \circ ” satisfying: for $u \in \mathcal{A}_i$, $v \in \mathcal{A}_j$ and $w \in \mathcal{A}_l$,

$$(u \circ v) \circ w = (-1)^{jl}(u \circ w) \circ v, \quad (u, v, w) = (-1)^{ij}(v, u, w), \quad (1.8)$$

where the associator:

$$(u, v, w) = (u \circ v) \circ w - u \circ (v \circ w). \quad (1.9)$$

When $\mathcal{A}_1 = \{0\}$, we call \mathcal{A} a *Novikov algebra*. It was essentially stated in [GDo1] that a quadratic homogeneous conformal algebra is equivalent to a Novikov algebra. Such an algebraic structure appeared in [BN] from the point of view of Poisson structures of

hydrodynamic type. The name “Novikov algebra” was given by Osborn [O1]. The above superanalogue was given in [X6].

By *quadratic conformal superalgebra*, we mean a conformal superalgebra R that is a free $\mathbb{C}[\partial]$ -module over its \mathbb{Z}_2 -graded subspace V such that for $u, v \in V$,

$$Y^+(u, z)v = (w_1 + \partial w_2)z^{-1} + w_3z^{-2} \quad \text{with } w_i \in V. \quad (1.10)$$

It was essentially stated in [GDo1] (without proof) that a quadratic conformal superalgebra is equivalent to a bialgebraic structure $(\mathcal{A}, [\cdot, \cdot], \circ)$ such that $(\mathcal{A}, [\cdot, \cdot])$ forms a Lie algebra, (\mathcal{A}, \circ) forms a Novikov algebra and the following compatibility condition holds:

$$[w \circ u, v] - [w \circ v, u] + [w, u] \circ v - [w, v] \circ u - w \circ [u, v] = 0 \quad (1.11)$$

for $u, v, w \in \mathcal{A}$. We may call such a bialgebraic structure a *Gel'fand-Dorfman* bialgebra for convenience. A quadratic conformal algebra corresponds to a Hamiltonian pair in [GDo1], which plays fundamental roles in completely integrable systems. It was also pointed out in [GDo1] that if we define the commutator

$$[u, v]^- = u \circ v - v \circ u \quad (1.12)$$

for a Novikov algebra (\mathcal{A}, \circ) , then $(\mathcal{A}, [\cdot, \cdot]^-, \circ)$ forms a Gel'fand-Dorfman bialgebra.

In this paper, we shall study quadratic conformal superalgebras. Naturally, we need the following concept. A *Super Gel'fand-Dorfman bialgebra* is a \mathbb{Z}_2 -graded vector space $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with two algebraic operations $[\cdot, \cdot]$ and \circ such that $(\mathcal{A}, [\cdot, \cdot])$ forms a Lie superalgebra, (\mathcal{A}, \circ) forms a Novikov superalgebra and the following compatibility condition holds:

$$[w \circ u, v] - (-1)^{ij}[w \circ v, u] + [w, u] \circ v - (-1)^{ij}[w, v] \circ u - w \circ [u, v] = 0 \quad (1.13)$$

for $u \in \mathcal{A}_i$, $v \in \mathcal{A}_j$ and $w \in \mathcal{A}$. We shall present the proof of that a quadratic conformal superalgebra is equivalent to a super Gel'fand-Dorfman bialgebra. For a Novikov superalgebra (\mathcal{A}, \circ) , we define another operation $[\cdot, \cdot]^-$ on \mathcal{A} by

$$[u, v]^- = u \circ v - (-1)^{ij}v \circ u \quad \text{for } u \in \mathcal{A}_i, v \in \mathcal{A}_j. \quad (1.14)$$

The proof of the fact that $(\mathcal{A}, [\cdot, \cdot]^-, \circ)$ forms a super Gel'fand-Dorfman bialgebra will also be given. However, we do not claim these proofs as our major results in this paper. Our purpose of doing these is to give the reader a convenience.

Our main results can be divided into two aspects. First, we shall present four general constructions of super Gel'fand-Dorfman bialgebras. Two of them are extracted from

simple Lie superalgebras of Cartan types W, H and K. One construction comes from a family of infinite-dimensional simple Lie superalgebras that we constructed in [X4]. The other construction is obtained from our classification works in this paper.

The second aspect of our results are classifications of Gel'fand-Dorfman bialgebras related to simple Novikov algebras. Zelmanov [Z] proved that any finite-dimensional simple Novikov algebra over an algebraically closed field with characteristic 0 is one-dimensional. Osborn [O2] classified finite-dimensional simple Novikov algebras with an idempotent element over an algebraically closed field with prime characteristic. In [X3], we gave a complete classification of finite-dimensional simple Novikov algebras and their irreducible modules over an algebraically closed field with prime characteristic.

Suppose that Δ is an additive subgroup of \mathbb{C} and denote $\Gamma = \{0\}$ or $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers. Let $\mathcal{A}_{\Delta, \Gamma}$ be a vector space with a basis $\{x_{\alpha, j} \mid (\alpha, j) \in \Delta \times \Gamma\}$. For any given constant $b \in \mathbb{C}$, we define algebraic operation \circ on $\mathcal{A}_{\Delta, \Gamma}$ by

$$x_{\alpha, i} \circ x_{\beta, j} = (\beta + b)x_{\alpha+\beta, i+j} + jx_{\alpha+\beta, i+j-1} \quad \text{for } \alpha, \beta \in \Delta, i, j \in \Gamma, \quad (1.15)$$

where we adopt the convention that if a notion is not defined but technically appears in an expression, we always treat it as zero; for instance, $x_{\alpha, j} = 0$ if $(\alpha, j) \notin \Delta \times \Gamma$ (this convention will be used throughout this paper). In [O3], Osborn proved that a simple Novikov algebras with an idempotent element whose left multiplication operator is locally-finite over an algebraically closed field with characteristic 0 must be isomorphic to $(\mathcal{A}_{\Delta, \Gamma}, \circ)$ for some Δ, Γ and b . There is a natural commutative associative algebra structure \cdot on $\mathcal{A}_{\Delta, \Gamma}$:

$$x_{\alpha, i} \cdot x_{\beta, j} = x_{\alpha+\beta, i+j} \quad \text{for } \alpha, \beta \in \Delta, i, j \in \Gamma. \quad (1.16)$$

Throughout this paper, the symbol \cdot of an associative algebraic operation in a product will be invisible for convenience, when the context is clear. For any $\xi \in \mathcal{A}_{\Delta, \Gamma}$, we define an algebraic operation \diamond_{ξ} on $\mathcal{A}_{\Delta, \Gamma}$ by

$$x_{\alpha, i} \diamond_{\xi} x_{\beta, j} = (\beta + \xi)x_{\alpha+\beta, i+j} + jx_{\alpha+\beta, i+j-1} \quad \text{for } \alpha, \beta \in \Delta, i, j \in \Gamma. \quad (1.17)$$

We proved in [X4] that $(\mathcal{A}_{\Delta, \Gamma}, \diamond_{\xi})$ forms a simple Novikov algebra for any $\xi \in \mathcal{A}_{\Delta, \Gamma}$. This in particular gives a large family of simple Novikov algebras without any idempotent elements. According to Gel'fand and Dorfman's statement in [GDo1], we have a large family of Gel'fand-Dorfman bialgebra $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]^{-}, \diamond_{\xi})$ (cf. (1.12)). In fact,

$$[x_{\alpha, i}, x_{\beta, j}]^{-} = x_{\alpha, i} \diamond_{\xi} x_{\beta, j} - x_{\beta, j} \diamond_{\xi} x_{\alpha, i} = (\beta - \alpha)x_{\alpha+\beta, i+j} + (j - i)x_{\alpha+\beta, i+j-1} \quad (1.18)$$

for $\alpha, \beta \in \Delta$, $i, j \in \Gamma$. Moreover, $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]^-)$ is a simple Lie algebra (cf. [O5]).

For convenience, we call $(\mathcal{A}, [\cdot, \cdot])$ a *Lie superalgebra over the Novikov superalgebra* (\mathcal{A}, \circ) and (\mathcal{A}, \circ) a *Novikov superalgebra over the Lie superalgebra* $(\mathcal{A}, [\cdot, \cdot])$ when $(\mathcal{A}, [\cdot, \cdot], \circ)$ forms a super Gel'fand-Dorfman bialgebra. In [OZ], the authors proved that a Lie algebra over $(\mathcal{A}_{\Delta, \Gamma}, \circ)$ with $\Delta = \mathbb{Z}$, $\Gamma = \{0\}$ and $b \notin \mathbb{Z}$ or $\Delta = \{0\}$ and $\Gamma = \mathbb{N}$ must be isomorphic to $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot]^-)$. In this paper, we shall classify all the Lie algebras over $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ for arbitrary additive subgroup Δ and any constant b . In the case $b \notin \Delta$, the Lie algebras are Block algebras [B]. We also classify the Lie algebras over $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ when $b \notin \Delta$. It seems to us that there are too many complicated Lie algebras over $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ when $\Delta \neq \{0\}$ and $b \in \Delta$. We shall present several families of such Lie algebras which still look neat. Furthermore, we shall classify all the Novikov algebras whose commutator algebra is $(\mathcal{A}_{\Delta, \{0\}}, [\cdot, \cdot]^-)$. It looks more challenging to classify Novikov superalgebras over all the well-known simple Lie superalgebras.

We remark that the Hamiltonian superoperator corresponding to a conformal superalgebra R with $R_1 \neq \{0\}$ does not in general have any analytic implication yet. The right theory of Hamiltonian operators compatible with supersymmetric partial differential equations is that we gave in [X6].

The paper is organized as follows. In Section 2, we mainly present the proof of the equivalence of a quadratic conformal superalgebra and a super Gel'fand-Dorfman bialgebra. In Section 3, we give four general constructions of super Gel'fand-Dorfman bialgebras. Our classification results are presented in Sections 4, 5 and 6.

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2 Equivalence

In this section, we shall prove that quadratic conformal superalgebras are equivalent to super Gel'fand-Dorfman bialgebras. Moreover, we shall also give the proof of that $(\mathcal{A}, [\cdot, \cdot]^-)$ forms a super Gel'fand-Dorfman bialgebra for any Novikov superalgebra (\mathcal{A}, \circ) (cf. (1.14)).

Let R be a \mathbb{Z}_2 -graded free $\mathbb{C}[\partial]$ -module over its \mathbb{Z}_2 -graded subspace V ($\partial(R_i) \subset R_i$). Let $Y^+(\cdot, z) : V \rightarrow LM(V, R[z^{-1}]z^{-1})$ be any given \mathbb{Z}_2 -graded linear map. We can extend $Y^+(\cdot, z)$ as follows. First we extend $Y^+(\cdot, z)$ to a linear map from R to $LM(V, R[z^{-1}])$:

$$Y^+(f(\partial)u, z)v = f(d/dz)Y^+(u, z)v \quad \text{for } u, v \in V, f(\partial) \in \mathbb{C}[\partial]. \quad (2.1)$$

Then we define a linear map $Y^+(\cdot, z) : R \rightarrow LM(R, R[z^{-1}])$ by

$$Y^+(\zeta, z)\partial^m v = \sum_{j=0}^m (-1)^j \partial^{m-j} d^j Y^+(\zeta, z)v/dz^j \quad \text{for } \zeta \in R, v \in V, m \in \mathbb{N}. \quad (2.2)$$

The extended map $Y^+(\cdot, z)$ naturally satisfies (1.2). According to Remark 4.1.2 (2) and (4.2.7) in [X7], $(R, Y^+(\cdot, z))$ forms a conformal superalgebra if and only if the map $Y^+(\cdot, z)$ satisfies (1.3) and (1.4) when acting on V for $u, v \in V$. This fact was showed by Kac [Ka2] through a relatively more complicated approach.

The map is said to be homogeneous of degree m if it satisfies (1.6). Suppose that the map $Y^+(\cdot, z)$ can be written as:

$$Y^+(\cdot, z) = Y_1^+(\cdot, z) + Y_2^+(\cdot, z) + \cdots + Y_p^+(\cdot, z), \quad (2.3)$$

where all $Y_j^+(\cdot, z) : V \rightarrow LM(V, R[z^{-1}])$ are homogeneous \mathbb{Z}_2 -graded linear maps of different degrees and they are supposed to be extended as $Y^+(\cdot, z)$. The following lemma can be proved by comparing the degrees in (1.3) and (1.4).

Lemma 2.1. *The map $Y^+(\cdot, z)$ satisfies (1.3) if and only if all $Y_j^+(\cdot, z)$ satisfy (1.3). When $p = 2$, the map $Y^+(\cdot, z)$ satisfies (1.4) if and only if $Y_1(\cdot, z)$ and $Y_2(\cdot, z)$ satisfy (1.4), and the following condition holds:*

$$\begin{aligned} & (Y_1^+(u, z_1)Y_2^+(v, z_2) + Y_2^+(u, z_1)Y_1^+(v, z_2))w \\ & - (-1)^{ij}(Y_1^+(v, z_2)Y_2^+(u, z_1) + Y_2^+(v, z_2)Y_1^+(u, z_1))w \\ = & \text{Res}_x \frac{(Y_1^+(Y_2^+(u, z_1 - x)v, x) + Y_2^+(Y_1^+(u, z_1 - x)v, x))w}{z_2 - x} \end{aligned} \quad (2.4)$$

for $u \in V_i, v \in V_j$ and $w \in V$. In particular, if the family $(R, \partial, Y^+(\cdot, z))$ forms a conformal superalgebra, then both $(R, \partial, Y_1^+(\cdot, z))$ and $(R, \partial, Y_2^+(\cdot, z))$ form conformal superalgebras when $p = 2$.

Theorem 2.2. *A quadratic conformal superalgebra is equivalent to a super Gel'fand-Dorfman bialgebra.*

Proof. Let $(R, \partial, Y^+(\cdot, z))$ be a quadratic conformal superalgebra. By (1.10), we can write:

$$Y^+(u, z)v = (\partial(v \circ u) + [v, u])z^{-1} + v \cdot uz^{-2} \quad \text{for } u, v \in V, \quad (2.5)$$

where $\circ, [\cdot, \cdot], \cdot$ are three algebraic operations on V and the reason of changing the order of u and v on the right hand side is because we want our results consistent with the

notions of Novikov algebra and Gel'fand-Dorfman bialgebra (otherwise, we would obtain their "opposite algebras"). For $u \in V_i$ and $v \in V_j$, (1.3) becomes:

$$\begin{aligned} & (\partial(u \circ v) + [u, v])z^{-1} + u \cdot vz^{-2} \\ = & (-1)^{ij}[(\partial(v \cdot u - v \circ u) - [v, u])z^{-1} + v \cdot uz^{-2}], \end{aligned} \quad (2.6)$$

where $f(x) \in R[x]$. Since R is a free $\mathbb{C}[\partial]$ -module over V , we obtain:

$$v \cdot u = v \circ u + (-1)^{ij}u \circ v, \quad [u, v] = -(-1)^{ij}[v, u]. \quad (2.7)$$

Define two linear maps $Y_1^+(\cdot, z), Y_2^+(\cdot, z) : V \rightarrow LM(V, R[z^{-1}])$ by:

$$Y_1^+(u, z)v = [v, u]z^{-1}, \quad Y_2^+(u, z)v = \partial v \circ uz^{-1} + (v \circ u + (-1)^{ij}u \circ v)z^{-2} \quad (2.8)$$

for $u \in V_i, v \in V_j$. Then $Y_j^+(\cdot, z)$ is a homogeneous map of degree j and $Y^+(\cdot, z) = Y_1^+(\cdot, z) + Y_2^+(\cdot, z)$. According to the above lemma, both $Y_1^+(\cdot, z)$ and $Y_2^+(\cdot, z)$ satisfy (1.4). Let $u \in V_i, v \in V_j$. By (1.4) for $Y_1^+(\cdot, z)$, we have:

$$([w, v], u) - (-1)^{ij}[[w, u], v]z_1^{-1}z_2^{-1} = [w, [v, u]]z_1^{-1}z_2^{-1} \quad (2.9)$$

by Definition 2.7b in [Ka2], which implies the second equation in (1.7). Thus $(V, [\cdot, \cdot])$ forms a Lie superalgebra.

Next we let $u \in V_i, v \in V_j$ and $w \in V_l$. By (2.2) and Definition 2.7b in [Ka2],

$$\begin{aligned} & Y_2^+(u, z_1)Y_2^+(v, z_2)w \\ = & \partial^2(w \circ v) \circ uz_1^{-1}z_2^{-1} + \partial((w \circ v) \circ u + (-1)^{jl}(v \circ w) \circ u)z_1^{-1}z_2^{-2} \\ & + \partial[2(w \circ v) \circ u + (-1)^{i(j+l)}u \circ (w \circ v)]z_1^{-2}z_2^{-1} \\ & + [(w \circ v + (-1)^{jl}v \circ w) \circ u + (-1)^{i(j+l)}u \circ (w \circ v + (-1)^{jl}v \circ w)]z_1^{-2}z_2^{-2} \\ & + 2((w \circ v) \circ u + (-1)^{i(j+l)}u \circ (w \circ v))z_1^{-3}z_2^{-1}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \text{Res}_x \frac{Y_2^+(Y_2^+(u, z_1 - x)v, x)w}{z_2 - x} \\ = & -\partial w \circ (v \circ u)z_1^{-1}z_2^{-2} - 2(w \circ (v \circ u) + (-1)^{l(i+j)}(v \circ u) \circ w)z_1^{-1}z_2^{-3} + (-w \circ (v \circ u) \\ & + (-1)^{ij}w \circ (u \circ v) - (-1)^{l(i+j)}(v \circ u) \circ w + (-1)^{l(i+j)+ij}(u \circ v) \circ w)z_1^{-2}z_2^{-2} \\ & + (-1)^{ij}\partial w \circ (u \circ v)z_1^{-2}z_2^{-1} + (-1)^{ij}2(w \circ (u \circ v) \\ & + (-1)^{l(i+j)}(u \circ v) \circ w)z_1^{-3}z_2^{-1} \end{aligned} \quad (2.11)$$

by (2.1). Thus by (1.4),

$$\begin{aligned} & \partial^2[(w \circ v) \circ u - (-1)^{ij}(w \circ u) \circ v]z_1^{-1}z_2^{-1} - (-1)^{ij}2((w \circ u) \circ v \\ & + (-1)^{j(i+l)}v \circ (w \circ u))z_1^{-1}z_2^{-3} + \partial[(w \circ v) \circ u + (-1)^{jl}(v \circ w) \circ u \\ & - (-1)^{ij}2(w \circ u) \circ v + (-1)^{j(i+l)}v \circ (w \circ u))]z_1^{-1}z_2^{-2} - (-1)^{ij}\partial[(w \circ u) \circ v \end{aligned}$$

$$\begin{aligned}
& +(-1)^{il}(u \circ w) \circ v - (-1)^{ij}(2(w \circ v) \circ u + (-1)^{i(j+l)}u \circ (w \circ v))z_1^{-2}z_2^{-1} \\
& + \{[(w \circ v + (-1)^{jl}v \circ w) \circ u + (-1)^{i(j+l)}u \circ (w \circ v + (-1)^{jl}v \circ w)] \\
& - (-1)^{ij}[(w \circ u + (-1)^{il}u \circ w) \circ v + (-1)^{j(i+l)}v \circ (w \circ u \\
& + (-1)^{il}u \circ w)]\}z_1^{-2}z_2^{-2} + 2((w \circ v) \circ u + (-1)^{i(j+l)}u \circ (w \circ v))z_1^{-3}z_2^{-1} \\
= & Y_2^+(u, z_1)Y_2^+(v, z_2)w - (-1)^{ij}Y_2^+(v, z_2)Y_2^+(u, z_1)w \\
= & \text{Res}_x \frac{Y_2^+(Y_2^+(u, z_1 - x)v, x)w}{z_2 - x} \\
= & -\partial w \circ (v \circ u)z_1^{-1}z_2^{-2} - 2(w \circ (v \circ u) + (-1)^{l(i+j)}(v \circ u) \circ w)z_1^{-1}z_2^{-3} + (-w \circ (v \circ u) \\
& + (-1)^{ij}w \circ (u \circ v) - (-1)^{l(i+j)}(v \circ u) \circ w + (-1)^{l(i+j)+ij}(u \circ v) \circ w)z_1^{-2}z_2^{-2} \\
& + (-1)^{ij}\partial w \circ (u \circ v)z_1^{-2}z_2^{-1} + (-1)^{ij}2(w \circ (u \circ v) \\
& + (-1)^{l(i+j)}(u \circ v) \circ w)z_1^{-3}z_2^{-1}. \tag{2.12}
\end{aligned}$$

Comparing the coefficients of $z_1^{-1}z_2^{-1}$ in (2.12), we have:

$$(w \circ v) \circ u = (-1)^{ij}(w \circ u) \circ v. \tag{2.13}$$

From the coefficients of $z_1^{-1}z_2^{-2}$ in (2.12), we find

$$\begin{aligned}
& (w \circ v) \circ u + (-1)^{jl}(v \circ w) \circ u - (-1)^{ij}(2(w \circ u) \circ v + (-1)^{j(i+l)}v \circ (w \circ u)) \\
= & -w \circ (v \circ u), \tag{2.14}
\end{aligned}$$

which is equivalent to:

$$(-1)^{jl}(v \circ w) \circ u - (w \circ v) \circ u - (-1)^{jl}v \circ (w \circ u) = -w \circ (v \circ u) \tag{2.15}$$

by (2.13). Note that (2.15) can be written as:

$$(w, v, u) = (-1)^{jl}(v, w, u) \tag{2.16}$$

(cf. (1.9)). Hence (V, \circ) forms a Novikov superalgebra.

We want to show that the coefficients of the other monomials in (2.12) do not give more constraints on (V, \circ) . Checking the coefficient of $z_1^{-1}z_2^{-3}$, we have:

$$-(-1)^{ij}2((w \circ u) \circ v + (-1)^{j(i+l)}v \circ (w \circ u)) = -2(w \circ (v \circ u) + (-1)^{l(i+j)}(v \circ u) \circ w), \tag{2.17}$$

which is equivalent to:

$$(w \circ v) \circ u + (-1)^{jl}v \circ (w \circ u) = w \circ (v \circ u) + (-1)^{lj}(v \circ w) \circ u \tag{2.18}$$

by (2.13). Note (2.18) is the same as (2.15). The equation from the coefficient of $z_1^{-2}z_2^{-1}$ is equivalent to that from the coefficient of $z_1^{-1}z_2^{-2}$ and the equation from the coefficient of $z_1^{-3}z_2^{-1}$ is equivalent to that from the coefficient of $z_1^{-1}z_2^{-3}$ because u, v, w are arbitrary. Extracting the coefficient of $z_1^{-2}z_2^{-2}$ in (2.12), we obtain:

$$\begin{aligned}
& (w \circ v + (-1)^{jl}v \circ w) \circ u + (-1)^{i(j+l)}u \circ (w \circ v + (-1)^{jl}v \circ w) \\
& - (-1)^{ij}[(w \circ u + (-1)^{il}u \circ w) \circ v + (-1)^{j(i+l)}v \circ (w \circ u + (-1)^{il}u \circ w)] \\
= & -w \circ (v \circ u) + (-1)^{ij}w \circ (u \circ v) - (-1)^{l(i+j)}(v \circ u) \circ w \\
& + (-1)^{l(i+j)+ij}(u \circ v) \circ w,
\end{aligned} \tag{2.19}$$

which is equivalent to:

$$\begin{aligned}
& (-1)^{jl}(v, w, u) - (-1)^{i(j+l)}(u, w, v) \\
= & (w, v, u) - (-1)^{ij}(w, u, v) - (-1)^{l(i+j)}(v, u, w) + (-1)^{ij+l(i+j)}(u, v, w).
\end{aligned} \tag{2.20}$$

The above equation is implied by (2.16) again because u, v, w are arbitrary.

Now we want to show that (2.4) only give rise to (1.13). First we have:

$$\begin{aligned}
& (Y_1^+(u, z_1)Y_2^+(v, z_2) + Y_2^+(u, z_1)Y_1^+(v, z_2))w \\
= & \partial([w \circ v, u] + [w, v] \circ u)z_1^{-1}z_2^{-1} + [w \circ v + (-1)^{jl}v \circ w, u]z_1^{-1}z_2^{-2} \\
& + ([w, v] \circ u + (-1)^{i(j+l)}u \circ [w, v] + [w \circ v, u])z_1^{-2}z_2^{-1},
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
& \text{Res}_x(z_2 - x)^{-1}[Y_1^+(Y_2^+(u, z_1 - x)v, x)w + Y_2^+(Y_1^+(u, z_1 - x)v, x)w] \\
= & \partial w \circ [v, u]z_1^{-1}z_2^{-1} + (w \circ [v, u] + (-1)^{l(i+j)}[v, u] \circ w - [w, v \circ u])z_1^{-1}z_2^{-2} \\
& + (w \circ [v, u] + (-1)^{l(i+j)}[v, u] \circ w + (-1)^{ij}[w, u \circ v])z_1^{-2}z_2^{-1}.
\end{aligned} \tag{2.22}$$

Then by (2.4), we get

$$\begin{aligned}
& \partial([w \circ v, u] + [w, v] \circ u - (-1)^{ij}([w \circ u, v] + [w, u] \circ v))z_1^{-1}z_2^{-1} \\
& + ([w \circ v + (-1)^{jl}v \circ w, u] - (-1)^{ij}([w, u] \circ v + (-1)^{j(i+l)}v \circ [w, u] + [w \circ u, v]))z_1^{-1}z_2^{-2} \\
& + ([w, v] \circ u + (-1)^{i(j+l)}u \circ [w, v] + [w \circ v, u] - (-1)^{ij}[w \circ u + (-1)^{jl}u \circ w, v])z_1^{-2}z_2^{-1} \\
= & (Y_1^+(u, z_1)Y_2^+(v, z_2) + Y_2^+(u, z_1)Y_1^+(v, z_2))w \\
& - (-1)^{ij}(Y_1^+(v, z_2)Y_2^+(u, z_1) + Y_2^+(v, z_1)Y_1^+(u, z_1))w \\
= & \text{Res}_x(z_2 - x)^{-1}[Y_1^+(Y_2^+(u, z_1 - x)v, x)w + Y_2^+(Y_1^+(u, z_1 - x)v, x)w] \\
= & \partial w \circ [v, u]z_1^{-1}z_2^{-1} + (w \circ [v, u] + (-1)^{l(i+j)}[v, u] \circ w - [w, v \circ u])z_1^{-1}z_2^{-2} \\
& + (w \circ [v, u] + (-1)^{l(i+j)}[v, u] \circ w + (-1)^{ij}[w, u \circ v])z_1^{-2}z_2^{-1}.
\end{aligned} \tag{2.23}$$

Note that the coefficients of $z_1^{-1}z_2^{-1}$ in (2.23) imply:

$$[w \circ v, u] + [w, v] \circ u - (-1)^{ij}([w \circ u, v] + [w, u] \circ v) = w \circ [v, u], \tag{2.24}$$

which is the same as (1.13) because u, v, w are arbitrary. Comparing the coefficients of $z_1^{-1}z_2^{-2}$ in (2.23), we obtain:

$$\begin{aligned}
& [w \circ v + (-1)^{jl} v \circ w, u] - (-1)^{ij} ([w, u] \circ v + (-1)^{j(i+l)} v \circ [w, u] + [w \circ u, v]) \\
= & w \circ [v, u] + (-1)^{l(i+j)} [v, u] \circ w - [w, v \circ u],
\end{aligned} \tag{2.25}$$

which can be rewritten as:

$$\begin{aligned}
& ([w \circ v, u] - (-1)^{ij} [w \circ u, v] - (-1)^{ij} [w, u] \circ v - w \circ [v, u] + [w, v] \circ u) \\
& + ((-1)^{jl} [v, w] \circ u + (-1)^{jl} [v \circ w, u] - (-1)^{l(i+j)} [v \circ u, w] \\
& - (-1)^{l(i+j)} [v, u] \circ w - (-1)^{jl} v \circ [w, u]) = 0.
\end{aligned} \tag{2.26}$$

However, (2.26) is implied by (2.24) because u, v, w are arbitrary. The equation from the coefficients of $z_1^{-2} z_2^{-1}$ in (2.23) is equivalent to (2.25), again because u, v, w are arbitrary.

What we have proved in the above is that $(V, [\cdot, \cdot], \circ)$ forms a super Gel'fand-Dorman bialgebra. Since the above arguments are reversible, a super Gel'fand-Dorman bialgebra $(V, [\cdot, \cdot], \circ)$ defines a quadratic conformal superalgebra by (2.5) and the first equation in (2.7). Thus the equivalence is established. \square

Theorem 2.3. *Let (\mathcal{A}, \circ) be a Novikov superalgebra. Then $(\mathcal{A}, [\cdot, \cdot]^-, \circ)$ (cf. (1.14)) forms a super Gel'fand-Dorfman bialgebra.*

Proof. For $u \in \mathcal{A}_i$, $v \in \mathcal{A}_j$ and $w \in \mathcal{A}_l$,

$$\begin{aligned}
& [w \circ u, v]^- - (-1)^{ij} [w \circ v, u]^- + [w, u]^- \circ v - (-1)^{ij} [w, v]^- \circ u - w \circ [u, v]^- \\
= & (w \circ u) \circ v - (-1)^{j(i+l)} v \circ (w \circ u) - (-1)^{ij} (w \circ v) \circ u + (-1)^{il} u \circ (w \circ v) \\
& + (w \circ u) \circ v - (-1)^{il} (u \circ w) \circ v - (-1)^{ij} (w \circ v) \circ u \\
& + (-1)^{(i+l)j} (v \circ w) \circ u - w \circ (u \circ v) + (-1)^{ij} w \circ (v \circ u) \\
= & -(-1)^{j(i+l)} v \circ (w \circ u) + (-1)^{il} u \circ (w \circ v) - (-1)^{il} (u \circ w) \circ v \\
& + (-1)^{(i+l)j} (v \circ w) \circ u - w \circ (u \circ v) + (-1)^{ij} w \circ (v \circ u) \\
= & (-1)^{(i+l)j} (v, w, u) - (-1)^{il} (u, w, v) - w \circ (u \circ v) + (-1)^{ij} w \circ (v \circ u) \\
= & (-1)^{ij} (w, v, u) - (w, u, v) - w \circ (u \circ v) + (-1)^{ij} w \circ (v \circ u) \\
= & (-1)^{ij} (w \circ v) \circ u - (-1)^{ij} w \circ (v \circ u) - (w \circ u) \circ v + w \circ (u \circ v) \\
& - w \circ (u \circ v) + (-1)^{ij} w \circ (v \circ u) = 0
\end{aligned} \tag{2.27}$$

by (1.8). So (1.13) holds. \square

Example. A super commutative associative algebra \mathcal{A} is \mathbb{Z}_2 -graded associative algebra such that

$$u \cdot v = (-1)^{ij} v \cdot u \quad \text{for } u \in \mathcal{A}_i, v \in \mathcal{A}_j. \quad (2.28)$$

It is easily verified that a super commutative associative algebra forms a Novikov super-algebra.

An element $d \in \text{End } \mathcal{A}$ is called a *derivation* of \mathcal{A} if there exists $i \in \mathbb{Z}_2$ such that

$$d(\mathcal{A}_j) \subset \mathcal{A}_{i+j}, \quad d(u \cdot v) = d(u) \cdot v + (-1)^{ij} v \cdot d(v), \quad (2.29)$$

for $j \in \mathbb{Z}_2$, $u \in \mathcal{A}_j$, $v \in \mathcal{A}$. The derivation d is called *even* if $i = 0$ and is called *odd* if $i = 1$.

Let d be an even derivation of a super commutative associative algebra \mathcal{A} and $\xi \in \mathcal{A}_0$. We define an operation \circ on \mathcal{A} by

$$u \circ v = ud(v) + \xi uv \quad \text{for } u, v \in \mathcal{A}. \quad (2.30)$$

By the analogous arguments as (2.7) and (2.8) in [X4], we can prove that (\mathcal{A}, \circ) forms a Novikov superalgebra.

3 Constructions

In this section, we shall give four constructions of super Gel'fand-Dorfman bialgebras.

Let (\mathcal{A}, \cdot) be a super commutative associative algebra. Denote by $W(\mathcal{A})_0$ the space of even derivations of \mathcal{A} and by $W(\mathcal{A})_1$ the space of odd derivations of \mathcal{A} . Then

$$W(\mathcal{A}) = W(\mathcal{A})_0 + W(\mathcal{A})_1 \quad (3.1)$$

forms a Lie superalgebra with respect to $[\cdot, \cdot]$ defined by

$$[d_1, d_2](v) = d_1(d_2(v)) - (-1)^{ij} d_2(d_1(v)) \quad \text{for } d_1 \in W(\mathcal{A})_i, d_2 \in W(\mathcal{A})_j, v \in \mathcal{A}. \quad (3.2)$$

Moreover, $W(\mathcal{A})$ forms a left \mathcal{A} -module with respect to the action:

$$(ad)(v) = ad(v) \quad \text{for } a, v \in \mathcal{A}, d \in W(\mathcal{A}). \quad (3.3)$$

Set

$$\mathcal{N} = W(\mathcal{A}) \oplus \mathcal{A}. \quad (3.4)$$

We define two algebraic operation $[\cdot, \cdot]$ and \circ on \mathcal{N} by

$$[d_1 + \xi_1, d_2 + \xi_2] = [d_1, d_2] + d_1(\xi_2) - (-1)^{ij} d_2(\xi_1), \quad (3.5)$$

$$(d_1 + \xi_1) \circ (d_2 + \xi_2) = (-1)^{ij} \xi_2 d_1 + \xi_1 \xi_2 \quad (3.6)$$

for $d_1 + \xi_1 \in \mathcal{N}_i = W(\mathcal{A})_i + \mathcal{A}_i$ and $d_2 + \xi_2 \in \mathcal{N}_j$.

Theorem 3.1. *The family $(\mathcal{N}, [\cdot, \cdot], \circ)$ forms a super Gel'fand-Dorfman bialgebra.*

Proof. The pair $(\mathcal{N}, [\cdot, \cdot])$ forms a Lie superalgebra because it is a semi-product of $W(\mathcal{A})$ with its module \mathcal{A} . Let $d_i + \xi_i \in \mathcal{N}_{j_i}$ with $i = 1, 2, 3$. First we have:

$$\begin{aligned} [(d_1 + \xi_1) \circ (d_2 + \xi_2)] \circ (d_3 + \xi_3) &= (-1)^{j_1 j_2 + (j_1 + j_2) j_3} \xi_3 \xi_2 d_1 + \xi_1 \xi_2 \xi_3 \\ &= (-1)^{j_2 j_3} [(d_1 + \xi_1) \circ (d_3 + \xi_3)] \circ (d_2 + \xi_2), \end{aligned} \quad (3.7)$$

$$\begin{aligned} (d_1 + \xi_1) \circ [(d_2 + \xi_2) \circ (d_3 + \xi_3)] &= (d_1 + \xi_1) \circ ((-1)^{j_2 j_3} \xi_3 d_2 + \xi_2 \xi_3) \\ &= (-1)^{j_1 (j_2 + j_3)} \xi_2 \xi_3 d_1 + \xi_1 \xi_2 \xi_3. \end{aligned} \quad (3.8)$$

The above two expressions imply the associativity:

$$[(d_1 + \xi_1) \circ (d_2 + \xi_2)] \circ (d_3 + \xi_3) = (d_1 + \xi_1) \circ [(d_2 + \xi_2) \circ (d_3 + \xi_3)]. \quad (3.9)$$

Thus (\mathcal{N}, \circ) forms a Novikov superalgebra. Furthermore,

$$\begin{aligned} &[(d_3 + \xi_3) \circ (d_1 + \xi_1), d_2 + \xi_2] - (-1)^{j_1 j_2} [(d_3 + \xi_3) \circ (d_2 + \xi_2), d_1 + \xi_1] \\ &+ [d_3 + \xi_3, d_1 + \xi_1] \circ (d_2 + \xi_2) - (-1)^{j_1 j_2} [d_3 + \xi_3, d_2 + \xi_2] \circ (d_1 + \xi_1) \\ &- (d_3 + \xi_3) \circ [d_1 + \xi_1, d_2 + \xi_2] \\ = &[(-1)^{j_1 j_3} \xi_1 d_3 + \xi_3 \xi_1, d_2 + \xi_2] - (-1)^{j_1 j_2} [(-1)^{j_2 j_3} \xi_2 d_3 + \xi_3 \xi_2, d_1 + \xi_1] \\ &+ ([d_3, d_1] + d_3(\xi_1) - (-1)^{j_1 j_3} d_1(\xi_3)) \circ (d_2 + \xi_2) - (-1)^{j_1 j_2} ([d_3, d_2] + d_3(\xi_2) \\ &- (-1)^{j_2 j_3} d_2(\xi_3)) \circ (d_1 + \xi_1) - (d_3 + \xi_3) \circ ([d_1, d_2] + d_1(\xi_2) - (-1)^{j_1 j_2} d_2(\xi_1)) \\ = &(-1)^{j_1 j_3} \xi_1 [d_3, d_2] - (-1)^{j_1 j_3 + j_2 (j_1 + j_3)} d_2(\xi_1) d_3 + (-1)^{j_1 j_3} \xi_1 d_3(\xi_2) \\ &- (-1)^{j_2 (j_1 + j_3)} d_2(\xi_3 \xi_1) - (-1)^{j_2 (j_1 + j_3)} \xi_2 [d_3, d_1] + (-1)^{(j_1 + j_2) j_3} d_1(\xi_2) d_3 \\ &- (-1)^{j_2 (j_1 + j_3)} \xi_2 d_3(\xi_1) + (-1)^{j_1 j_3} d_1(\xi_3 \xi_2) + (-1)^{j_2 (j_1 + j_3)} \xi_2 [d_3, d_1] \\ &+ (d_3(\xi_1) - (-1)^{j_1 j_3} d_1(\xi_3)) \xi_2 - (-1)^{j_1 j_3} \xi_1 [d_3, d_2] - (-1)^{j_1 j_2} (d_3(\xi_2) \\ &- (-1)^{j_2 j_3} d_2(\xi_3)) \xi_1 - (-1)^{j_3 (j_1 + j_2)} d_1(\xi_2) d_3 + (-1)^{j_1 j_2 + j_3 (j_1 + j_2)} d_2(\xi_1) d_3 \\ &- \xi_3 d_1(\xi_2) + (-1)^{j_1 j_2} \xi_3 d_2(\xi_1) = 0. \quad \square \end{aligned} \quad (3.10)$$

The above construction is extracted from the simple Lie superalgebras of Cartan type W (cf. [Ka3], [X7]).

Our second and third constructions are related to the following concept. A *Lie-Poisson superalgebra* \mathcal{A} is a \mathbb{Z}_2 -graded space with two algebraic operations \cdot and $[\cdot, \cdot]$ such that (\mathcal{A}, \cdot) forms a super commutative associative algebra, $(\mathcal{A}, [\cdot, \cdot])$ forms a Lie superalgebra and the following compatibility condition is satisfied:

$$[u, v \cdot w] = [u, v] \cdot w + (-1)^{ij} v \cdot [u, w] \quad \text{for } u \in \mathcal{A}_i, v \in \mathcal{A}_j, w \in \mathcal{A}. \quad (3.11)$$

Let $(\mathcal{A}, \cdot, [\cdot, \cdot])$ be a Lie-Poisson superalgebra and let d be an even derivation of the algebra (\mathcal{A}, \cdot) such that

$$d[u, v] = [d(u), v] + [u, d(v)] + \xi[u, v] \quad \text{for } u, v \in \mathcal{A}, \quad (3.12)$$

where $\xi \in \mathbb{F}$ is a constant. Now we define another algebraic operation \circ on \mathcal{A} by

$$u \circ v = ud(v) + \xi uv \quad \text{for } u \in \mathcal{A}, v \in \mathcal{A}_\beta. \quad (3.13)$$

Theorem 3.2. *The family $(\mathcal{A}, [\cdot, \cdot], \circ)$ forms a Gel'fand-Dorfman super bialgebra.*

Proof. Note that (3.13) is the same as the second equation in (2.15) in [X4]. It can be similarly verified that (\mathcal{A}, \circ) forms a Novikov superalgebra. Moreover, for $u \in \mathcal{A}_i, v \in \mathcal{A}_j$ and $w \in \mathcal{A}$,

$$\begin{aligned} & [w \circ u, v] - (-1)^{ij} [w \circ v, u] + [w, u] \circ v - (-1)^{ij} [w, v] \circ u - w \circ [u, v] \\ = & [w(d + \xi)(u), v] - (-1)^{ij} [w(d + \xi)(v), u] + [w, u](d + \xi)(v) \\ & - (-1)^{ij} [w, v](d + \xi)(u) - w(d + \xi)[u, v] \\ = & [wd(u), v] - (-1)^{ij} [wd(v), u] + [w, u]d(v) - (-1)^{ij} [w, v]d(u) + \xi([wu, v] \\ & - (-1)^{ij} [wv, u] + [w, u]v - (-1)^{ij} [w, v]u) - w([d(u), v] + [u, d(v)] + 2\xi[u, v]) \\ = & (-1)^{ij} [w, v]d(u) - [w, u]d(v) + [w, u]d(v) - (-1)^{ij} [w, v]d(u) \\ & + \xi([w[u, v] - (-1)^{ij} w[v, u]] - w(2\xi[u, v])) = 0. \end{aligned} \quad (3.14)$$

So (1.13) holds. \square

The above construction is related to the Lie superalgebras of Hamiltonian type and Contact type (cf. [Ka1]).

Next we shall present a construction extracted from a family of infinite-dimensional simple Lie superalgebras that we obtained in [X4] (cf. Theorem 5.3 in [X4]).

Let (\mathcal{A}, \cdot) be a commutative associative algebra and let d be a derivation of \mathcal{A} . Set

$$\tilde{\mathcal{A}} = \mathcal{A} \times \mathcal{A} = \tilde{\mathcal{A}}_0 \oplus \tilde{\mathcal{A}}_1 \quad (3.15)$$

with

$$\tilde{\mathcal{A}}_0 = (\mathcal{A}, 0), \quad \tilde{\mathcal{A}}_1 = (0, \mathcal{A}). \quad (3.16)$$

For fixed elements $\xi, \eta_0, \eta_1 \in \mathcal{A}$, we define two algebraic operations $[\cdot, \cdot]$ and \circ on $\tilde{\mathcal{A}}$ by

$$[(u_0, u_1), (v_0, v_1)] = (\xi u_1 v_1, 0), \quad (3.17)$$

$$(u_0, u_1) \circ (v_0, v_1) = (u_0(d(v_0) + \eta_0 v_0), u_1(d(v_0) + \eta_0 v_0) + u_0(d(v_1) + \eta_1 v_1)) \quad (3.18)$$

for $u_i, v_j \in \mathcal{A}$. It is easily seen that $(\tilde{\mathcal{A}}, [\cdot, \cdot])$ forms a Lie superalgebra.

First we want to show that $(\tilde{\mathcal{A}}, \circ)$ forms a Novikov superalgebra. Note that by Corollary 2.6 in [X4], $(\tilde{\mathcal{A}}_0, \circ)$ forms a Novikov algebra. This fact will simplify our verification of (1.13). Let $u_i, v_i, w_i \in \mathcal{A}$ with $i = 0, 1$. First we have:

$$\begin{aligned} [(0, w_1) \circ (u_0, 0)] \circ (v_0, 0) &= (0, w_1(d(u_0) + \eta_0 u_0)) \circ (v_0, 0) \\ &= (0, w_1(d(u_0) + \eta_0 u_0)(d(v_0) + \eta_0 v_0)), \end{aligned} \quad (3.19)$$

$$\begin{aligned} [(w_0, w_1) \circ (0, u_1)] \circ (v_0, 0) &= (0, w_0(d(u_1) + \eta_1 u_1)) \circ (v_0, 0) \\ &= (0, w_0(d(u_1) + \eta_1 u_1)(d(v_0) + \eta_0 v_0)), \end{aligned} \quad (3.20)$$

$$\begin{aligned} [(w_0, w_1) \circ (v_0, 0)] \circ (0, u_1) &= (w_0(d(v_0) + \eta_0 v_0), w_1(d(v_0) + \eta_0 v_0)) \circ (0, u_1) \\ &= (0, w_0(d(v_0) + \eta_0 v_0)(d(u_1) + \eta_1 u_1)), \end{aligned} \quad (3.21)$$

$$[(w_0, w_1) \circ (0, u_1)] \circ (0, v_1) = (0, w_0(d(u_1) + \eta_1 u_1)) \circ (0, v_1) = (0, 0). \quad (3.22)$$

Thus the first equation in (1.8) holds. Furthermore,

$$\begin{aligned} &((u_0, 0), (v_0, 0), (0, w_1)) \\ &= [(u_0, 0) \circ (v_0, 0)] \circ (0, w_1) - (u_0, 0) \circ [(v_0, 0) \circ (0, w_1)] \\ &= (0, u_0(d(v_0) + \eta_0 v_0)(d(w_1) + \eta_1 w_1) - (0, u_0(d(v_0 d(w_1) + \eta_1 v_0 w_1) \\ &\quad + \eta_1(v_0 d(w_1) + \eta_1 v_0 w_1))) \\ &= (0, u_0 v_0[(\eta_0 - \eta_1)(d(w_1) + \eta_1 w_1) - d^2(w_1) - d(\eta_1 w_1)]), \end{aligned} \quad (3.23)$$

$$\begin{aligned} &((u_0, 0), (0, v_1), (w_0, w_1)) \\ &= [(u_0, 0) \circ (0, v_1)] \circ (w_0, w_1) - (u_0, 0) \circ [(0, v_1) \circ (w_0, w_1)] \\ &= (0, u_0(d(v_1) + \eta_1 v_1)(d(w_0) + \eta_0 w_0)) - (0, u_0(d(v_1 d(w_0) + \eta_0 v_1 w_0) \\ &\quad + \eta_1(v_1 d(w_0) + \eta_0 v_1 w_0))) \\ &= (0, -u_0 v_1 d(d(w_0) + \eta_0 w_0)), \end{aligned} \quad (3.24)$$

$$\begin{aligned}
& ((0, v_1), (u_0, 0), (w_0, w_1)) \\
&= [(0, v_1) \circ (u_0, 0)] \circ (w_0, w_1) - (0, v_1) \circ [(u_0, 0) \circ (w_0, w_1)] \\
&= (0, v_1(d(u_0) + \eta_0 u_0)(d(w_0) + \eta_0 w_0)) - (0, v_1(d(u_0 d(w_0) + \eta_0 u_0 w_0) \\
&\quad + \eta_0(u_0 d(w_0) + \eta_0 u_0 w_0))) \\
&= (0, -u_0 v_1 d(d(w_0) + \eta_0 w_0)), \tag{3.25}
\end{aligned}$$

$$((0, u_1), (0, v_1), (w_0, w_1)) = 0 = -((0, v_1), (0, u_1), (w_0, w_1)). \tag{3.26}$$

Expressions (3.23-26) show that the second equation in (1.8) holds. Hence $(\tilde{\mathcal{A}}, \circ)$ forms a Novikov superalgebra.

We shall now exam (1.13). Note (1.13) trivially holds if $u = (u_0, 0)$ and $v = (v_0, 0)$ by (3.17). Moreover,

$$\begin{aligned}
& [(w_0, w_1) \circ (u_0, 0), (0, v_1)] - [(w_0, w_1) \circ (0, v_1), (u_0, 0)] + [(w_0, w_1), (u_0, 0)] \circ (0, v_1) \\
& - [(w_0, w_1), (0, v_1)] \circ (u_0, 0) - (w_0, w_1) \circ [(u_0, 0), (0, v_1)] \\
&= (\xi w_1 v_1(d(u_0) + \eta_0 u_0), 0) - 0 + 0 - (\xi w_1 v_1(d(u_0) + \eta_0 u_0), 0) - 0 = 0, \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
& [(w_0, w_1) \circ (0, u_1), (0, v_1)] + [(w_0, w_1) \circ (0, v_1), (0, u_1)] \\
& + [(w_0, w_1), (0, u_1)] \circ (0, v_1) + [(w_0, w_1), (0, v_1)] \circ (0, u_1) - (w_0, w_1) \circ [(0, u_1), (0, v_1)] \\
&= (\xi w_0 v_1(d(u_1) + \eta_1 u_1), 0) + (\xi w_0 u_1(d(v_1) + \eta_1 v_1), 0) + (0, \xi w_1 u_1(d(v_1) + \eta_1 v_1)) \\
& + (0, \xi w_1 v_1(d(u_1) + \eta_1 u_1)) - (w_0(d(\xi u_1 v_1) + \eta_0 \xi u_1 v_1), w_1(d(\xi u_1 v_1) + \eta_0 \xi u_1 v_1)) \\
&= (w_0 u_1 v_1(2\xi \eta_1 - \xi \eta_0 - d(\xi)), w_1 u_1 v_1(2\xi \eta_1 - \xi \eta_0 - d(\xi))). \tag{3.28}
\end{aligned}$$

Therefore, (1.13) holds if and only if

$$2\xi \eta_1 = \xi \eta_0 + d(\xi). \tag{3.29}$$

We summarize the above result as the following theorem.

Theorem 3.3 *Let (\mathcal{A}, \cdot) be a commutative associative algebra and let d be a derivation of \mathcal{A} . We set a space $\tilde{\mathcal{A}}$ as in (3.15) and (3.16). For any given three elements $\xi, \eta_0, \eta_1 \in \mathcal{A}$ satisfying (3.29), we define two algebraic operations $[\cdot, \cdot]$ and \circ on $\tilde{\mathcal{A}}$ as in (3.17) and (3.18). Then the family $(\tilde{\mathcal{A}}, [\cdot, \cdot], \circ)$ forms a Gel'fand-Dorfman super bialgebra.*

Let (\mathcal{A}, \cdot) be a commutative associative algebra and let d_1, d_2, d_3 be mutually commutative derivations of (\mathcal{A}, \cdot) . Define

$$[u, v]_{i,j} = d_i(u)d_j(v) - d_j(u)d_i(v), \quad [u, v]_i = u d_i(v) - d_i(u)v \quad \text{for } u, v \in \mathcal{A}; \tag{3.30}$$

$$u \circ_{i,b} v = u(d_i + b)(v) \quad \text{for } u, v \in \mathcal{A}, \quad (3.31)$$

where $b \in \mathbb{F}$ is a constant. It can be verified that $(\mathcal{A}, [\cdot, \cdot]_{i,j})$ forms a Lie algebra. Moreover, $(\mathcal{A}, \circ_{i,b})$ form Novikov algebras by Corollary 2.6 in [X4]. We want to use these algebras to construct new Gel'fand-Dorfman bialgebras.

For $u, v, w \in \mathcal{A}$, we denote

$$\begin{aligned} [u, v, w]_{i,j,l} &= [[u, v]_{i,j}, w]_l + [[u, v]_l, w]_{i,j} + [[v, w]_{i,j}, u]_l \\ &+ [[v, w]_l, u]_{i,j} + [[w, u]_{i,j}, v]_l + [[w, u]_l, v]_{i,j}. \end{aligned} \quad (3.32)$$

A tedious calculation shows that

$$\begin{aligned} [u, v, w]_{i,j,l} &= d_i(u)d_l(v)d_j(w) + d_j(u)d_i(v)d_l(w) + d_l(u)d_j(v)d_i(w) \\ &- d_i(u)d_j(v)d_l(w) - d_j(u)d_l(v)d_i(w) - d_l(u)d_i(v)d_j(w) \end{aligned} \quad (3.33)$$

for $u, v, w \in \mathcal{A}$. By (3.33), we can verify that

$$[\cdot, \cdot, \cdot]_{i,j,j} = 0. \quad (3.34)$$

Thus we have:

Proposition 3.4. *The pair $(\mathcal{A}, [\cdot, \cdot]_{1,2} + \lambda[\cdot, \cdot]_1)$ form a Lie algebra for any $\lambda \in \mathbb{F}$.*

For $u, v, w \in \mathcal{A}$, we have:

$$\begin{aligned} &[w \circ_{3,b} u, v]_{1,2} - [w \circ_{3,b} v, u]_{1,2} + [w, u]_{1,2} \circ_{3,b} v - [w, v]_{1,2} \circ_{3,b} u - w \circ_{3,b} [u, v]_{1,2} \\ = &d_1(w)d_3(u)d_2(v) + wd_1d_3(u)d_2(v) + bd_1(w)ud_2(v) + bwd_1(u)d_2(v) \\ &- d_2(w)d_3(u)d_1(v) - wd_2d_3(u)d_1(v) - bd_2(w)ud_1(v) - bwd_2(u)d_1(v) \\ &- d_1(w)d_2(u)d_3(v) - wd_2(u)d_1d_3(v) - bd_1(w)d_2(u)v - bwd_2(u)d_1(v) \\ &+ d_2(w)d_1(u)d_3(v) + wd_1(u)d_2d_3(v) + bd_2(w)d_1(u)v + bwd_1(u)d_2(v) \\ &+ d_1(w)d_2(u)d_3(v) + bd_1(w)d_2(u)v - d_2(w)d_1(u)d_3(v) - bd_2(w)d_1(u)v \\ &- d_1(w)d_3(u)d_2(v) - bd_1(w)ud_2(v) + d_2(w)d_3(u)d_1(v) + bd_2(w)ud_1(v) \\ &- wd_1d_3(u)d_2(v) - wd_1(u)d_2d_3(v) - bwd_1(u)d_2(v) \\ &+ wd_2(u)d_1d_3(v) + wd_2d_3(u)d_1(v) + bwd_2(u)d_1(v) \\ = &bw[u, v]_{1,2}, \end{aligned} \quad (3.35)$$

$$\begin{aligned} &[w \circ_{2,b} u, v]_1 - [w \circ_{2,b} v, u]_1 + [w, u]_1 \circ_{2,b} v - [w, v]_1 \circ_{2,b} u - w \circ_{2,b} [u, v]_1 \\ = &wd_2(u)d_1(v) + bwd_1(v) - d_1(w)d_2(u)v - wd_1d_2(u)v - bd_1(w)uv - bwd_1(u)v \\ &- wd_1(u)d_2(v) - bwd_1(u)v + d_1(w)ud_2(v) + wud_1d_2(v) + bd_1(w)uv + bwd_1(v) \\ &+ wd_1(u)d_2(v) + bwd_1(u)v - d_1(w)ud_2(v) - bd_1(w)uv - wd_2(u)d_1(v) \end{aligned}$$

$$\begin{aligned}
& -bwud_1(v) + d_1(w)d_2(u)v + bd_1(w)uv - bwud_1(v) - wd_2(u)d_1(v) \\
& -wud_1d_2(v) + bwd_1(u)v + wd_1d_2(u)v + wd_1(u)d_2(v) \\
& = w[u, v]_{1,2}.
\end{aligned} \tag{3.36}$$

Proposition 3.4 and the above two expressions imply the following theorem.

Theorem 3.5. *The families $(\mathcal{A}, [\cdot, \cdot]_{1,2} + [\cdot, \cdot]_2, \circ_{2,0})$ and $(\mathcal{A}, [\cdot, \cdot]_{2,1} + b[\cdot, \cdot]_1, \circ_{2,b})$ form Gel'fand-Dorfman bialgebras.*

Remark 3.6. (a) Theorems 3.2 and 3.5 will be used in next sections of classifications.

(b) For a Lie-Poisson superalgebra $(\mathcal{A}, \cdot, [\cdot, \cdot])$, the family $(\mathcal{A}, [\cdot, \cdot], \cdot)$ in general does not satisfy (1.13). So it in general does not form a Gel'fand-Dorfman super bialgebra.

4 Classification I

In this section, we shall classify the Lie algebras over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ defined in (1.15).

For convenience, we redenote

$$x_\alpha = x_{\alpha,0} \quad \text{for } \alpha \in \Delta; \quad \mathcal{A} = \mathcal{A}_{\Delta, \{0\}}. \tag{4.1}$$

Now (1.15) becomes:

$$x_\alpha \circ x_\beta = (\beta + b)x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \tag{4.2}$$

Assume that $(\mathcal{A}, [\cdot, \cdot])$ is a Lie algebra over (\mathcal{A}, \circ) . Set

$$[x_\alpha, x_\beta] = \sum_{\sigma \in \Delta} a_{\alpha, \beta}^\sigma x_\sigma \quad \text{for } \alpha, \beta \in \Delta, \tag{4.3}$$

where $a_{\alpha, \beta}^\sigma \in \mathbb{C}$ are the structure constants. The skew-symmetry of Lie algebra implies:

$$a_{\alpha, \beta}^\sigma = -a_{\beta, \alpha}^\sigma \quad \text{for } \alpha, \beta, \sigma \in \Delta. \tag{4.4}$$

For any $\alpha, \beta, \gamma \in \Delta$, by (1.11) and (4.4),

$$\begin{aligned}
& [x_\gamma \circ x_\alpha, x_\beta] - [x_\gamma \circ x_\beta, x_\alpha] + [x_\gamma, x_\alpha] \circ x_\beta - [x_\gamma, x_\beta] \circ x_\alpha - x_\gamma \circ [x_\alpha, x_\beta] \\
& = \sum_{\sigma \in \Delta} \{[(\alpha + b)a_{\alpha+\gamma, \beta}^\sigma - (\beta + b)a_{\beta+\gamma, \alpha}^\sigma]x_\sigma + (\beta + b)a_{\gamma, \alpha}^\sigma x_{\sigma+\beta} - (\alpha + b)a_{\gamma, \beta}^\sigma x_{\sigma+\alpha} \\
& \quad - (\sigma + b)a_{\alpha, \beta}^\sigma x_{\sigma+\gamma}\} \\
& = \sum_{\sigma \in \Delta} [(\alpha + b)(a_{\alpha+\gamma, \beta}^\sigma - a_{\gamma, \beta}^{\sigma-\alpha}) + (\beta + b)(a_{\alpha, \beta+\gamma}^\sigma - a_{\alpha, \gamma}^{\sigma-\beta}) - (\sigma + b - \gamma)a_{\alpha, \beta}^{\sigma-\gamma}]x_\sigma \\
& = 0,
\end{aligned} \tag{4.5}$$

which is equivalent to:

$$(\alpha + b)(a_{\alpha+\gamma,\beta}^\sigma - a_{\gamma,\beta}^{\sigma-\alpha}) + (\beta + b)(a_{\alpha,\beta+\gamma}^\sigma - a_{\alpha,\gamma}^{\sigma-\beta}) - (\sigma + b - \gamma)a_{\alpha,\beta}^{\sigma-\gamma} = 0 \quad (4.6)$$

for $\alpha, \beta, \gamma, \sigma \in \Delta$.

Letting $\alpha = \gamma = 0$ in (4.6), we have:

$$(\beta + b)a_{0,\beta}^\sigma - (\sigma + b)a_{0,\beta}^\sigma = 0 \quad \text{for } \beta, \sigma \in \Delta, \quad (4.7)$$

which is equivalent to:

$$(\beta - \sigma)a_{0,\beta}^\sigma = 0 \quad \text{for } \beta, \sigma \in \Delta. \quad (4.8)$$

Thus

$$a_{0,\beta}^\sigma = 0 \quad \text{for } \beta, \sigma \in \Delta; \beta \neq \sigma. \quad (4.9)$$

We denote

$$\varphi(\beta) = a_{0,\beta}^\beta \quad \text{for } \beta \in \Delta. \quad (4.10)$$

Obviously $\varphi(0) = 0$ by (4.4). By (4.9),

$$[x_0, x_\beta] = \varphi(\beta)x_\beta \quad \text{for } \beta \in \Delta. \quad (4.11)$$

Letting $\alpha = 0$, $\sigma = \beta + \gamma$ in (4.6), we obtain:

$$(\beta + b)(a_{0,\beta+\gamma}^{\beta+\gamma} - a_{0,\gamma}^\gamma - a_{0,\beta}^\beta) = 0, \quad (4.12)$$

which implies:

$$\varphi(\beta + \gamma) = \varphi(\beta) + \varphi(\gamma) \quad \text{for } \beta, \gamma \in \Delta; \beta \neq -b \text{ or } \gamma \neq -b. \quad (4.13)$$

If $b \in \Delta$, we have

$$0 = \varphi(b + (-b)) = \varphi(b) + \varphi(-b). \quad (4.14)$$

Hence by (4.13) and (4.14),

$$\varphi(-b) + \varphi(-b) = \varphi(-b) + (\varphi(b) + \varphi(-2b)) = \varphi(-2b). \quad (4.15)$$

By (4.13) and (4.15), $\varphi : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism.

We define an operator \mathcal{D} on \mathcal{A} by:

$$\mathcal{D}(u) = x_0 \circ u \quad \text{for } u \in \mathcal{A}. \quad (4.16)$$

Then

$$\mathbb{C}x_\alpha = \{u \in \mathcal{A} \mid \mathcal{D}(u) = (\alpha + b)u\}. \quad (4.17)$$

Letting $w = x_0$, $u = x_\alpha$ and $v = x_\beta$ in (1.11) for $\alpha, \beta \in \Delta$, we get:

$$(\alpha + b)[x_\alpha, x_\beta] - (\beta + b)[x_\beta, x_\alpha] + \varphi(\alpha)(\beta + b)x_{\alpha+\beta} - \varphi(\beta)(\alpha + b)x_{\beta+\alpha} - \mathcal{D}[x_\alpha, x_\beta] = 0, \quad (4.18)$$

which is equivalent to:

$$(\mathcal{D} - \alpha - \beta - 2b)[x_\alpha, x_\beta] = (\varphi(\alpha)(\beta + b) - \varphi(\beta)(\alpha + b))x_{\alpha+\beta}. \quad (4.19)$$

If $b = 0$, then (4.17) and (4.19) imply:

$$[x_\alpha, x_\beta] = a_{\alpha, \beta}^{\alpha+\beta} x_{\alpha+\beta}, \quad (4.20)$$

and

$$\varphi(\alpha)\beta = \varphi(\beta)\alpha \quad (4.21)$$

for $\alpha, \beta \in \Delta$. When $\Delta = \{0\}$, the classification is trivial. So we assume $\Delta \neq \{0\}$. Let $\alpha_0 \in \Delta$ be any fixed nonzero element and set

$$a = \frac{\varphi(\alpha_0)}{a_0}. \quad (4.22)$$

By (4.21), we have:

$$\varphi(\beta) = a\beta \quad \text{for } \beta \in \Delta. \quad (4.23)$$

Set

$$\phi(\alpha, \beta) = a_{\alpha, \beta}^{\alpha+\beta} + a(\alpha - \beta) \quad \text{for } \alpha, \beta \in \Delta. \quad (4.24)$$

By (4.10) and (4.23),

$$\phi(0, \beta) = 0 \quad \text{for } \beta \in \Delta. \quad (4.25)$$

Moreover, (4.4) shows that ϕ is skew-symmetric. Letting $\sigma = \alpha + \beta + \gamma$ in (4.6) for $\alpha, \beta, \gamma \in \Delta$, we have:

$$\begin{aligned} & \alpha(\phi(\alpha + \gamma, \beta) + a(\beta - \alpha - \gamma) - \phi(\gamma, \beta) - a(\beta - \gamma)) \\ & + \beta(\phi(\alpha, \beta + \gamma) + a(\gamma + \beta - \alpha) - \phi(\alpha, \gamma) - a(\gamma - \alpha)) \\ & - (\alpha + \beta)(\phi(\alpha, \beta) + a(\beta - \alpha)) = 0, \end{aligned} \quad (4.26)$$

which is equivalent to:

$$\alpha(\phi(\alpha + \gamma, \beta) - \phi(\gamma, \beta) - \phi(\alpha, \beta)) = \beta(\phi(\beta + \gamma, \alpha) - \phi(\gamma, \alpha) - \phi(\beta, \alpha)). \quad (4.27)$$

For $\alpha, \beta, \gamma \in \Delta$, we set

$$S_0(\alpha, \beta, \gamma) = \begin{cases} 0 & \text{if } \alpha = 0, \\ \alpha^{-1}(\phi(\beta + \gamma, \alpha) - \phi(\gamma, \alpha) - \phi(\beta, \alpha)) & \text{if } \alpha \neq 0. \end{cases} \quad (4.28)$$

Note (4.25) and (4.27) imply that

$$S_0(\cdot, \cdot, \cdot) : \Delta \times \Delta \times \Delta \rightarrow \mathbb{C} \text{ is a symmetric map.} \quad (4.29)$$

Furthermore,

$$\phi(\beta + \gamma, \alpha) = \phi(\gamma, \alpha) + \phi(\beta, \alpha) + \alpha S_0(\alpha, \beta, \gamma) \quad \text{for } \alpha, \beta, \gamma \in \Delta. \quad (4.30)$$

So the map S_0 measures the nonlinearity of ϕ .

By (4.20) and (4.24), we have:

$$[x_\alpha, x_\beta] = (\phi(\alpha, \beta) + a(\beta - \alpha))x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.31)$$

Thus for α, β, γ , the Jacobi identity of Lie algebra imply:

$$\begin{aligned} & [[x_\alpha, x_\beta], x_\gamma] + [[x_\beta, x_\gamma], x_\alpha] + [[x_\gamma, x_\alpha], x_\beta] \\ = & (\phi(\alpha, \beta) + a(\beta - \alpha))(\phi(\alpha + \beta, \gamma) + a(\gamma - \alpha - \beta))x_{\alpha+\beta+\gamma} \\ & + (\phi(\beta, \gamma) + a(\gamma - \beta))(\phi(\beta + \gamma, \alpha) + a(\alpha - \beta - \gamma))x_{\alpha+\beta+\gamma} \\ & + (\phi(\gamma, \alpha) + a(\alpha - \gamma))(\phi(\gamma + \alpha, \beta) + a(\beta - \gamma - \alpha))x_{\alpha+\beta+\gamma} \\ = & (\phi(\alpha, \beta) + a(\beta - \alpha))(\phi(\alpha, \gamma) + \phi(\beta, \gamma) + \gamma S_0(\alpha, \beta, \gamma) + a(\gamma - \alpha - \beta))x_{\alpha+\beta+\gamma} \\ & + (\phi(\beta, \gamma) + a(\gamma - \beta))(\phi(\beta, \alpha) + \phi(\gamma, \alpha) + \alpha S_0(\alpha, \beta, \gamma) + a(\alpha - \beta - \gamma))x_{\alpha+\beta+\gamma} \\ & + (\phi(\gamma, \alpha) + a(\alpha - \gamma))(\phi(\gamma, \beta) + \phi(\alpha, \beta) + \beta S_0(\alpha, \beta, \gamma) + a(\beta - \gamma - \alpha))x_{\alpha+\beta+\gamma} \\ = & \{ \phi(\alpha, \beta)\phi(\alpha, \gamma) + \phi(\alpha, \beta)\phi(\beta, \gamma) + \phi(\beta, \gamma)\phi(\beta, \alpha) + \phi(\beta, \gamma)\phi(\gamma, \alpha) \\ & + \phi(\gamma, \alpha)\phi(\gamma, \beta) + \phi(\gamma, \alpha)\phi(\alpha, \beta) - a(\phi(\alpha, \beta)\gamma + \phi(\beta, \gamma)\alpha + \phi(\gamma, \alpha)\beta) + [\gamma\phi(\alpha, \beta) \\ & + a\gamma(\beta - \alpha) + \alpha\phi(\beta, \gamma) + a\alpha(\gamma - \beta) + \beta\phi(\gamma, \alpha) + a\beta(\alpha - \gamma)]S_0(\alpha, \beta, \gamma) \} x_{\alpha+\beta+\gamma} \\ = & (\gamma\phi(\alpha, \beta) + \alpha\phi(\beta, \gamma) + \beta\phi(\gamma, \alpha))(S_0(\alpha, \beta, \gamma) - a)x_{\alpha+\beta+\gamma} = 0, \end{aligned} \quad (4.32)$$

which is equivalent to

$$(\gamma\phi(\alpha, \beta) + \alpha\phi(\beta, \gamma) + \beta\phi(\gamma, \alpha))(S_0(\alpha, \beta, \gamma) - a) = 0. \quad (4.33)$$

Note the above arguments are reversible. We summarize the above result as the following theorem.

Theorem 4.1. *Any Lie algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $b = 0$ has its Lie bracket in the form (4.31), where a is a constant and $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ is a skew-symmetric map such that there exists a symmetric map $S_0(\cdot, \cdot, \cdot) : \Delta \times \Delta \times \Delta \rightarrow \mathbb{C}$ satisfying (4.30), (4.33). Conversely, for any constant $a \in \mathbb{C}$ and a skew-symmetric*

map $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ such that there exists a symmetric map $S_0(\cdot, \cdot, \cdot) : \Delta \times \Delta \times \Delta \rightarrow \mathbb{C}$ satisfying (4.30) and (4.33), the bracket in (4.31) define a Lie algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $b = 0$. In particular, the following ϕ satisfies our condition: when $a = 0$, ϕ is any skew-symmetric \mathbb{Z} -bilinear form; when $a \neq 0$,

$$\phi(\alpha, \beta) = \alpha\varphi_0(\beta) - \beta\varphi_0(\alpha) \quad \text{for } \alpha, \beta \in \Delta, \quad (4.34)$$

where $\varphi_0 : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism.

Next we consider the case $b \notin \Delta$. Now by (4.17) and (4.19), we have:

$$-b[x_\alpha, x_\beta] = (\varphi(\alpha)(\beta + b) - \varphi(\beta)(\alpha + b))x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.35)$$

Thus

$$[x_\alpha, x_\beta] = \frac{1}{b}(\varphi(\beta)\alpha - \varphi(\alpha)\beta + b(\varphi(\beta) - \varphi(\alpha)))x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.36)$$

Theorem 4.2. *A Lie algebra is a Lie algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $b \notin \Delta$ if and only if its Lie bracket has the form (4.36), where $\varphi : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism.*

Proof. We only need to prove sufficient part. Let $\varphi : \Delta \rightarrow \mathbb{C}^+$ be a group homomorphism and define the operation $[\cdot, \cdot]$ on $\mathcal{A} = \mathcal{A}_{\Delta, \{0\}}$ by (4.36). Moreover, we define two operations d_1 and d_2 on \mathcal{A} by:

$$d_1(x_\alpha) = \frac{1}{b}\varphi(\alpha)x_\alpha, \quad d_2(x_\alpha) = \alpha x_\alpha \quad \text{for } \alpha \in \Delta. \quad (4.37)$$

Then d_1 and d_2 are mutually commutative derivations of \mathcal{A} . By Theorem 3.6, $(\mathcal{A}, [\cdot, \cdot], \circ) = (\mathcal{A}, [\cdot, \cdot]_{2,1} + b[\cdot, \cdot]_1, \circ_{2,b})$ forms a Gel'fand-Dorfman bialgebra. \square

Finally we consider the case $0 \neq b \in \Delta$. Again by (4.17) and (4.19), we have:

$$[x_\alpha, x_\beta] = a_{\alpha, \beta}^{\alpha+\beta+b} x_{\alpha+\beta+b} + a_{\alpha, \beta}^{\alpha+\beta} x_{\alpha+\beta} \quad (4.38)$$

with

$$a_{\alpha, \beta}^{\alpha+\beta} = \frac{1}{b}(\varphi(\beta)\alpha - \varphi(\alpha)\beta + b(\varphi(\beta) - \varphi(\alpha))). \quad (4.39)$$

We set

$$\theta(\alpha, \beta) = a_{\alpha, \beta}^{\alpha+\beta+b} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.40)$$

By (4.4) and (4.9), $\theta(\cdot, \cdot) : \Delta \rightarrow \mathbb{C}$ is a skew-symmetric map and

$$\theta(0, \beta) = 0 \quad \text{for } \beta \in \Delta. \quad (4.41)$$

Letting $\sigma = \alpha + \beta + \gamma + b$ in (4.6) for $\alpha, \beta, \gamma \in \Delta$, we obtain:

$$(\alpha + b)(\theta(\alpha + \gamma, \beta) - \theta(\gamma, \beta) - \theta(\alpha, \beta)) = (\beta + b)(\theta(\beta + \gamma, \alpha) - \theta(\gamma, \alpha) - \theta(\beta, \alpha)). \quad (4.42)$$

In particular, for $\beta = -b$ and $\alpha \neq -b$ or $\gamma \neq -b$, we have:

$$\theta(\alpha + \gamma, -b) = \theta(\gamma, -b) + \theta(\alpha, -b). \quad (4.43)$$

As (4.13)-(4.15), we can prove that (4.43) holds for any $\alpha, \gamma \in \Delta$. We set

$$S_b(\alpha, \beta, \gamma) = \begin{cases} 0 & \text{if } \alpha = -b, \\ (\alpha + b)^{-1}(\theta(\beta + \gamma, \alpha) - \theta(\gamma, \alpha) - \theta(\beta, \alpha)) & \text{if } \alpha \neq -b \end{cases} \quad (4.44)$$

for $\alpha, \beta, \gamma \in \Delta$. Note (4.42) and (4.43) imply that

$$S_b(\alpha_1, \alpha_2, \alpha_3) \text{ is symmetric with respect to } \alpha_i \text{ and } \alpha_j \quad (4.45)$$

whenever $(\alpha_i + b)(\alpha_j + b) \neq 0$. Moreover,

$$\theta(\beta + \gamma, \alpha) = \theta(\gamma, \alpha) + \theta(\beta, \alpha) + (\alpha + b)S_b(\alpha, \beta, \gamma) \quad \text{for } \alpha, \beta, \gamma \in \Delta. \quad (4.46)$$

For α, β, γ , by (4.45) and (4.46), the Jacobi identity of Lie algebra imply:

$$\begin{aligned} & [[x_\alpha, x_\beta], x_\gamma] + [[x_\beta, x_\gamma], x_\alpha] + [[x_\gamma, x_\alpha], x_\beta] \\ = & \frac{1}{b} \{ [b\theta(\alpha, \beta)x_{\alpha+\beta+b} + ((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha))x_{\alpha+\beta}, x_\gamma] \\ & + [b\theta(\beta, \gamma)x_{\beta+\gamma+b} + ((\beta + b)\varphi(\gamma) - (\gamma + b)\varphi(\beta))x_{\beta+\gamma}, x_\alpha] \\ & + [b\theta(\gamma, \alpha)x_{\gamma+\alpha+b} + ((\gamma + b)\varphi(\alpha) - (\alpha + b)\varphi(\gamma))x_{\alpha+\gamma}, x_\beta] \} \\ = & [\theta(\alpha, \beta)\theta(\alpha + \beta + b, \gamma) + \theta(\beta, \gamma)\theta(\beta + \gamma + b, \alpha) + \theta(\gamma, \alpha)\theta(\gamma + \alpha + b, \beta)]x_{\alpha+\beta+\gamma+2b} \\ & + \frac{1}{b} [\theta(\alpha, \beta)((\alpha + \beta + 2b)\varphi(\gamma) - (\gamma + b)\varphi(\alpha + \beta + b)) + \theta(\alpha + \beta, \gamma)((\alpha + b)\varphi(\beta) \\ & - (\beta + b)\varphi(\alpha)) + \theta(\beta, \gamma)((\beta + \gamma + 2b)\varphi(\alpha) - (\alpha + b)\varphi(\beta + \gamma + b)) \\ & + \theta(\beta + \gamma, \alpha)((\beta + b)\varphi(\gamma) - (\gamma + b)\varphi(\beta)) + \theta(\gamma, \alpha)((\gamma + \alpha + 2b)\varphi(\beta) \\ & - (\beta + b)\varphi(\gamma + \alpha + b)) + \theta(\gamma + \alpha, \beta)((\gamma + b)\varphi(\alpha) - (\alpha + b)\varphi(\gamma))]x_{\alpha+\beta+\gamma+b} \\ = & [\theta(\alpha, \beta)\theta(\alpha + \beta + b, \gamma) + \theta(\beta, \gamma)\theta(\beta + \gamma + b, \alpha) + \theta(\gamma, \alpha)\theta(\gamma + \alpha + b, \beta)]x_{\alpha+\beta+\gamma+2b} \\ & - \frac{1}{b} [(\gamma + b)\theta(\alpha, \beta) + (\alpha + b)\theta(\beta, \gamma) + (\beta + b)\theta(\gamma, \alpha)]\varphi(b)x_{\alpha+\beta+\gamma+b} \\ = & 0, \end{aligned} \quad (4.47)$$

which is equivalent to:

$$\theta(\alpha, \beta)\theta(\alpha + \beta + b, \gamma) + \theta(\beta, \gamma)\theta(\beta + \gamma + b, \alpha) + \theta(\gamma, \alpha)\theta(\gamma + \alpha + b, \beta) = 0 \quad (4.48)$$

and

$$[(\gamma + b)\theta(\alpha, \beta) + (\alpha + b)\theta(\beta, \gamma) + (\beta + b)\theta(\gamma, \alpha)]\varphi(b) = 0. \quad (4.49)$$

When $\gamma = 0$ in (4.49), we get:

$$\theta(\alpha, \beta)\varphi(b) = 0 \quad \text{for } \alpha, \beta \in \Delta. \quad (4.50)$$

Thus (4.49) is equivalent to (4.50) and $\theta \equiv 0$ if $\varphi(b) \neq 0$.

If $\varphi(b) = 0$ and θ is \mathbb{Z} -bilinear, then (4.42) naturally holds and (4.48) is equivalent to

$$\theta(\alpha, \beta)\theta(b, \gamma) + \theta(\beta, \gamma)\theta(b, \alpha) + \theta(\gamma, \alpha)\theta(b, \beta) = 0 \quad \text{for } \alpha, \beta, \gamma \in \Delta \quad (4.51)$$

by the skew-symmetry of θ . If there exists $\gamma_0 \in \Delta$ such that $\theta(b, \gamma_0) \neq 0$, then

$$\theta(\alpha, \beta) = \theta(b, \alpha)\frac{\theta(\gamma_0, \beta)}{\theta(b, \gamma_0)} - \theta(b, \beta)\frac{\theta(\gamma_0, \alpha)}{\theta(b, \gamma_0)} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.52)$$

Thus the solution of (4.51) is:

$$\theta \text{ is any skew-symmetric } \mathbb{Z}\text{-bilinear form such that } b \in \text{Rad}_\theta \quad (4.53)$$

or

$$\theta(\alpha, \beta) = \varphi_1(\alpha)\varphi_2(\beta) - \varphi_1(\beta)\varphi_2(\alpha) \quad \text{for } \alpha, \beta \in \Delta, \quad (4.54)$$

where $\varphi_1, \varphi_2 : \Delta \rightarrow \mathbb{C}^+$ are group homomorphisms such that

$$\varphi_1(b) = 0, \quad \varphi_2(b) = -1. \quad (4.55)$$

Since the above arguments are reversible, we have the following theorem:

Theorem 4.3. *Any Lie algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $0 \neq b \in \Delta$ has its Lie bracket as follows:*

$$[x_\alpha, x_\beta] = \theta(\alpha, \beta)x_{\alpha+\beta+b} + \frac{1}{b}((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha))x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.56)$$

where $\varphi : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism, $\theta \equiv 0$ if $\varphi(b) \neq 0$ and $\theta : \Delta \times \Delta \rightarrow \mathbb{C}$ is a skew-symmetric map satisfying (4.42) and (4.48) if $\varphi(b) = 0$. Conversely, for any given group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$ and a skew-symmetric map $\theta : \Delta \times \Delta \rightarrow \mathbb{C}$ such that $\theta \equiv 0$ if $\varphi(b) \neq 0$ and (4.42), (4.48) hold if $\varphi(b) = 0$, (4.56) defines a Lie algebra

over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $0 \neq b \in \Delta$. In particular, for any group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$ such that $\varphi(b) = 0$ and a skew-symmetric \mathbb{Z} -bilinear map $\theta : \Delta \times \Delta \rightarrow \mathbb{C}$ in (4.53) or (4.54), (4.56) defines a Lie algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ with $0 \neq b \in \Delta$.

Remark 4.4. The following Lie algebraic structure on $\mathcal{A} = \mathcal{A}_{\Delta, \{0\}}$ seems interesting itself, although it not directly related to Gel'fand-Dorfman bialgebras. In fact, it can also be viewed as generalizations of Block algebras (cf. [B]). Assume that $0 \neq b \in \Delta$. Let $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ be a skew-symmetric \mathbb{Z} -bilinear form such that $b \notin \text{Rad}_\phi$ and let $\varphi : \Delta \rightarrow \mathbb{C}^+$ be a group homomorphism. We have the following Lie bracket on \mathcal{A} :

$$[x_\alpha, x_\beta] = (\varphi(\alpha)\phi(b, \beta) - \varphi(\beta)\phi(b, \alpha)x_{\alpha+\beta+b} + \phi(\alpha, \beta)x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.57)$$

5 Classification II

In this section, we shall classify the Lie algebras over the simple Novikov algebra $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ defined in (1.15) with $b \notin \Delta$. Several families of Lie algebras over the simple Novikov algebra $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ with $b \in \Delta$ will be constructed.

We start with the general case of arbitrary b . Let $(\mathcal{A}_{\Delta, \mathbb{N}}, [\cdot, \cdot])$ be a Lie algebra over the Novikov algebra $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$. Set

$$[x_{\alpha, i}, x_{\beta, j}] = \sum_{\sigma \in \Delta, k \in \mathbb{N}} a_{\alpha, i; \beta, j}^{\sigma, k} x_{\sigma, k} \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}. \quad (5.1)$$

Then the skew-symmetry of Lie algebra shows

$$a_{\alpha, i; \beta, j}^{\sigma, k} = -a_{\beta, j; \alpha, i}^{\sigma, k} \quad \text{for } \alpha, \beta, \sigma \in \Delta, i, j, k \in \mathbb{N}. \quad (5.2)$$

For $\alpha, \beta, \gamma \in \Delta$ and $i, j, l \in \mathbb{N}$, by (1.11) and (1.15),

$$\begin{aligned} & [x_{\gamma, l} \circ x_{\alpha, i}, x_{\beta, j}] - [x_{\gamma, l} \circ x_{\beta, j}, x_{\alpha, i}] + [x_{\gamma, l}, x_{\alpha, i}] \circ x_{\beta, j} \\ & - [x_{\gamma, l}, x_{\beta, j}] \circ x_{\alpha, i} - x_{\gamma, l} \circ [x_{\alpha, i}, x_{\beta, j}] \\ & = \sum_{\sigma \in \Delta, k \in \mathbb{N}} \{(\alpha + b)(a_{\alpha+\gamma, i+l; \beta, j}^{\sigma, k} x_{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma, k} x_{\sigma+\alpha, k+i}) + (\beta + b)(a_{\alpha, i; \beta+\gamma, j+l}^{\sigma, k} x_{\sigma, k} \\ & + a_{\gamma, l; \alpha, i}^{\sigma, k} x_{\sigma+\beta, k+j}) + i(a_{\alpha+\gamma, i+l-1; \beta, j}^{\sigma, k} x_{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma, k} x_{\sigma+\alpha, k+i-1}) + j(a_{\alpha, i; \beta+\gamma, j+l-1}^{\sigma, k} x_{\sigma, k} \\ & + a_{\gamma, l; \alpha, i}^{\sigma, k} x_{\sigma+\beta, k+j-1}) - (\sigma + b)a_{\alpha, i; \beta, j}^{\sigma, k} x_{\sigma+\gamma, k+l} - ka_{\alpha, i; \beta, j}^{\sigma, k} x_{\sigma+\gamma, k+l-1}\} \\ & = \sum_{\sigma \in \Delta, k \in \mathbb{N}} \{(\alpha + b)(a_{\alpha+\gamma, i+l; \beta, j}^{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma-\alpha, k-i}) + (\beta + b)(a_{\alpha, i; \beta+\gamma, j+l}^{\sigma, k} + a_{\gamma, l; \alpha, i}^{\sigma-\beta, k-j}) \\ & + i(a_{\alpha+\gamma, i+l-1; \beta, j}^{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma-\alpha, k+1-i}) + j(a_{\alpha, i; \beta+\gamma, j+l-1}^{\sigma, k} + a_{\gamma, l; \alpha, i}^{\sigma-\beta, k+1-j}) \\ & - (\sigma - \gamma + b)a_{\alpha, i; \beta, j}^{\sigma-\gamma, k-l} - (k+1-l)a_{\alpha, i; \beta, j}^{\sigma-\gamma, k+1-l}\} x_{\sigma, k} = 0, \end{aligned} \quad (5.3)$$

which is equivalent to:

$$\begin{aligned}
& (\alpha + b)(a_{\alpha+\gamma, i+l; \beta, j}^{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma-\alpha, k-i}) + (\beta + b)(a_{\alpha, i; \beta+\gamma, j+l}^{\sigma, k} + a_{\gamma, l; \alpha, i}^{\sigma-\beta, k-j}) \\
& + i(a_{\alpha+\gamma, i+l-1; \beta, j}^{\sigma, k} - a_{\gamma, l; \beta, j}^{\sigma-\alpha, k+1-i}) + j(a_{\alpha, i; \beta+\gamma, j+l-1}^{\sigma, k} + a_{\gamma, l; \alpha, i}^{\sigma-\beta, k+1-j}) \\
& - (\sigma - \gamma + b)a_{\alpha, i; \beta, j}^{\sigma-\gamma, k-l} - (k + 1 - l)a_{\alpha, i; \beta, j}^{\sigma-\gamma, k+1-l} = 0
\end{aligned} \tag{5.4}$$

for $\alpha, \beta, \gamma, \sigma \in \Delta$ and $i, j, l, k \in \mathbb{N}$.

Letting $\alpha = \gamma = 0$ and $i = l = 0$ in (5.4), we have:

$$(\beta + b)a_{0,0;\beta,j}^{\sigma,k} + ja_{0,0;\beta,j-1}^{\sigma,k} - (\sigma + b)a_{0,0;\beta,j}^{\sigma,k} - (k + 1)a_{0,0;\beta,j}^{\sigma,k+1} = 0, \tag{5.5}$$

that is,

$$a_{0,0;\beta,j}^{\sigma,k+1} = \frac{1}{k+1}[(\beta - \sigma)a_{0,0;\beta,j}^{\sigma,k} + ja_{0,0;\beta,j-1}^{\sigma,k}]. \tag{5.6}$$

Thus by mathematical induction, we have:

$$a_{0,0;\beta,j}^{\sigma,k} = \sum_{p=0}^k \frac{1}{(k-p)!} \binom{j}{p} (\beta - \sigma)^{k-p} a_{0,0;\beta,j-p}^{\sigma,0} \quad \text{for } \beta \in \Delta, j, k \in \mathbb{N}. \tag{5.7}$$

For any given $\beta \in \Delta$ and $j \in \mathbb{N}$, since $[x_{0,0}, x_{\beta,j}]$ is a finite linear combination of $\{x_{\sigma,k} \mid \sigma \in \Delta, k \in \mathbb{N}\}$, there exists integer $K > j$ such that

$$a_{0,0;\beta,j}^{\sigma,k} = 0 \quad \text{when } k \geq K. \tag{5.8}$$

Hence (5.7) and (5.8) give the following system:

$$\sum_{p=0}^j \frac{1}{(K+q-p)!} \binom{j}{p} (\beta - \sigma)^{K+q-p} a_{0,0;\beta,j-p}^{\sigma,0} = 0 \quad \text{for } q = 0, 1, 2, \dots, j. \tag{5.9}$$

Assume $\beta \neq \sigma$. We view $\{a_{0,0;\beta,j}^{\sigma,0}, a_{0,0;\beta,j-1}^{\sigma,0}, \dots, a_{0,0;\beta,0}^{\sigma,0}\}$ as unknowns. The coefficient determinant of the system (5.9) is:

$$\begin{aligned}
& \begin{vmatrix} \frac{(\beta-\sigma)^K}{K!}, & \frac{(\beta-\sigma)^{K-1}}{(K-1)!} \binom{j}{1} & \cdots, & \frac{(\beta-\sigma)^{K-j}}{(K-j)!} \binom{j}{j} \\ \frac{(\beta-\sigma)^{K+1}}{(K+1)!}, & \frac{(\beta-\sigma)^K}{K!} \binom{j}{1} & \cdots, & \frac{(\beta-\sigma)^{K+1-j}}{(K+1-j)!} \binom{j}{j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(\beta-\sigma)^{K+j}}{(K+j)!}, & \frac{(\beta-\sigma)^{K+j-1}}{(K+j-1)!} \binom{j}{1} & \cdots, & \frac{(\beta-\sigma)^K}{K!} \binom{j}{j} \end{vmatrix} \\
& = \left(\prod_{p=0}^j \binom{j}{p} \frac{(\beta-\sigma)^K}{(K+p)!} \right) \begin{vmatrix} 1, & K, & \cdots, & K(K-1) \cdots (K+1-j) \\ 1, & (K+1), & \cdots, & (K+1)K \cdots (K+2-j) \\ \vdots & \vdots & \ddots & \vdots \\ 1, & (K+j), & \cdots, & (K+j)(K+j-1) \cdots (K+1) \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \left[\left(\prod_{p=0}^j \binom{j}{p} \frac{(\beta - \sigma)^K}{(K+p)!} \left(\frac{d}{dz_p} \right)^p \right) \begin{vmatrix} 1, & z_1^K, & \dots, & z_j^K \\ 1, & z_1^{K+1}, & \dots, & z_j^{K+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1, & z_1^{K+j}, & \dots, & z_j^{K+j} \end{vmatrix} \right]_{z_1=z_2=\dots=z_j=1} \\
&= \left[\left(\prod_{p=0}^j \binom{j}{p} \frac{(\beta - \sigma)^K}{(K+p)!} \left(\frac{d}{dz_p} \right)^p \right) (z_1 z_2 \dots z_j)^K \begin{vmatrix} 1, & 1, & \dots, & 1 \\ 1, & z_1, & \dots, & z_j \\ \vdots & \vdots & \ddots & \vdots \\ 1, & z_1^j, & \dots, & z_j^j \end{vmatrix} \right]_{z_1=z_2=\dots=z_j=1} \\
&= \left[\left(\prod_{p=0}^j \binom{j}{p} \frac{(\beta - \sigma)^K}{(K+p)!} \left(\frac{d}{dz_p} \right)^p \right) (z_1 z_2 \dots z_j)^K \left(\prod_{i=1}^j (1 - z_i) \right) \left(\prod_{1 \leq s < t \leq j} (z_s - z_t) \right) \right]_{z_1=z_2=\dots=z_j=1} \\
&= (-1)^{j(j+1)/2} \prod_{p=0}^j \binom{j}{p} \frac{(\beta - \sigma)^K}{(K+p)!} \neq 0. \tag{5.10}
\end{aligned}$$

Thus we have:

$$a_{0,0;\beta,j}^{\sigma,0} = a_{0,0;\beta,j-1}^{\sigma,0} = \dots = a_{0,0;\beta,0}^{\sigma,0} = 0. \tag{5.11}$$

By (5.7), we obtain:

$$a_{0,0;\beta,j}^{\sigma,k} = 0 \quad \text{for } \beta \neq \sigma \in \Delta, k \in \mathbb{Z} \tag{5.12}$$

and

$$a_{0,0;\beta,j}^{\beta,k} = \begin{cases} 0 & \text{if } k > j, \\ \binom{j}{k} a_{0,0;\beta,j-k}^{\beta,0} & \text{if } k \leq j \end{cases} \tag{5.13}$$

Letting $\alpha = \gamma = 0$, $\sigma = \beta$, $i = 0$ and $k = l = 1$ in (5.4), we get:

$$(\beta + b)(a_{0,0;\beta,j+1}^{\beta,1} + a_{0,1;0,0}^{0,1-j}) + j(a_{0,0;\beta,j}^{\beta,1} + a_{0,1;0,0}^{0,2-j}) - (\beta + b)a_{0,0;\beta,j}^{\beta,0} - a_{0,0;\beta,j}^{\beta,1} = 0. \tag{5.14}$$

By (5.13) and (5.14),

$$(\beta + b)(ja_{0,0;\beta,j}^{\beta,0} + \delta_{1,j}a_{0,1;0,0}^{0,0}) + (j-1)ja_{0,0;\beta,j-1}^{\beta,0} + j\delta_{2,j}a_{0,1;0,0}^{0,0} = 0. \tag{5.15}$$

When $j = 1$ in (5.15), we have:

$$(\beta + b)(a_{0,0;\beta,1}^{\beta,0} + a_{0,1;0,0}^{0,0}) = 0. \tag{5.16}$$

Thus

$$a_{0,0;\beta,1}^{\beta,0} = a_{0,0;0,1}^{0,0} \quad \text{for } -b \neq \beta \in \Delta \tag{5.17}$$

by (5.2). When $j = 2$ in (5.15),

$$2(\beta + b)a_{0,0;\beta,2}^{\beta,0} + 2a_{0,0;\beta,1}^{\beta,0} + 2a_{0,1;0,0}^{0,0} = 0, \tag{5.18}$$

which is equivalent to:

$$a_{0,0;\beta,2}^{\beta,0} = 0 \quad \text{for } -b \neq \beta \in \Delta \tag{5.20}$$

by (5.2) and (5.17). When $j > 2$ in (5.15), we get:

$$j(\beta + b)a_{0,0;\beta,j}^{\beta,0} + (j-1)ja_{0,0;\beta,j-1}^{\beta,0} = 0, \quad (5.21)$$

which is equivalent to

$$(\beta + b)a_{0,0;\beta,j}^{\beta,0} = (1-j)a_{0,0;\beta,j-1}^{\beta,0}. \quad (5.22)$$

Moreover, by (5.20), (5.22) and mathematical induction on j , we can prove:

$$a_{0,0;\beta,j}^{\beta,0} = 0 \quad \text{for } -b \neq \beta \in \Delta, 2 \leq j \in \mathbb{N}. \quad (5.23)$$

When $\beta = -b$ in (5.15), we have:

$$(j-1)ja_{0,0;-b,j-1}^{-b,0} + j\delta_{2,j}a_{0,1;0,0}^{0,0} = 0, \quad (5.24)$$

which implies:

$$a_{0,0;-b,1}^{-b,0} = a_{0,0;0,1}^{0,0}, \quad a_{0,0;-b,j}^{-b,0} = 0 \quad \text{for } 2 \leq j \in \mathbb{N}. \quad (5.25)$$

Therefore, we have:

$$a_{0,0;\beta,1}^{\beta,0} = a_{0,0;0,1}^{0,0}, \quad a_{0,0;\beta,j}^{\beta,0} = 0 \quad \text{for } \beta \in \Delta, 2 \leq j \in \mathbb{N}. \quad (5.26)$$

We denote

$$a_{0,0;0,1}^{0,0} = \lambda, \quad a_{0,0;\beta,0}^{\beta,0} = \varphi(\beta) \quad \text{for } \beta \in \Delta. \quad (5.27)$$

By (5.13) and (5.26),

$$a_{0,0;\beta,j}^{\beta,j-1} = \lambda j \quad \text{for } \beta \in \Delta, j \in \mathbb{N}. \quad (5.28)$$

Thus we have:

$$[x_{0,0}, x_{\beta,j}] = \varphi(\beta)x_{\beta,j} + \lambda jx_{\beta,j-1} \quad \text{for } \beta \in \Delta, j \in \mathbb{N}. \quad (5.29)$$

Note that letting $\alpha = 0$, $\sigma = \beta + \gamma$ and $i = j = l = k = 0$ in (5.4), we get the same equation as (4.12) with $a_{0,\rho}^\rho$ replaced by $a_{0,0;\rho,0}^{\rho,0}$ for $\rho \in \Delta$ because $a_{0,0;\beta,0}^{\beta,1} = 0$ by (5.13). So $\varphi : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism by (4.12)-(4.15).

Define an operator \mathcal{D} on $\mathcal{A}_{\Delta,\mathbb{N}}$ by

$$\mathcal{D}(u) = x_{0,0} \circ u \quad \text{for } u \in \mathcal{A}_{\Delta,\mathbb{N}}. \quad (5.30)$$

Then

$$\sum_{p=0}^{j-1} \mathbb{C}x_{\beta,p} = \{u \in \mathcal{A}_{\Delta,\mathbb{N}} \mid (\mathcal{D} - \beta - b)^j(u) = 0\} \quad \text{for } \beta \in \Delta, j \in \mathbb{N}. \quad (5.31)$$

For $\alpha, \beta \in \Delta$ and $i, j \in \mathbb{N}$, letting $w = x_{0,0}$, $u = x_{\alpha,i}$ and $v = x_{\beta,j}$ in (1.11), we get:

$$\begin{aligned} & [(\alpha + b)x_{\alpha,i} + ix_{\alpha,i-1}, x_{\beta,j}] - [(\beta + b)x_{\beta,j} + jx_{\beta,j-1}, x_{\alpha,i}] + (\varphi(\alpha)x_{\alpha,i} + \lambda ix_{\alpha,i-1}) \circ x_{\beta,j} \\ & - (\varphi(\beta)x_{\beta,j} + \lambda jx_{\beta,j-1}) \circ x_{\alpha,i} - x_{0,0} \circ [x_{\alpha,i}, x_{\beta,j}] = 0, \end{aligned} \quad (5.32)$$

which is equivalent to:

$$\begin{aligned} & (\mathcal{D} - \alpha - \beta - 2b)([x_{\alpha,i}, x_{\beta,j}]) \\ & = i[x_{\alpha,i-1}, x_{\beta,j}] + j[x_{\alpha,i}, x_{\beta,j-1}] + (\varphi(\alpha)(\beta + b) - \varphi(\beta)(\alpha + b))x_{\alpha+\beta,i+j} \\ & + [i(\lambda(\beta + b) - \varphi(\beta)) + j(\varphi(\alpha) - \lambda(\alpha + b))]x_{\alpha+\beta,i+j-1}. \end{aligned} \quad (5.33)$$

Assume that $b \neq 0$. By (5.31), (5.33) and mathematical induction on $i + j$, we can prove that

$$[x_{\alpha,i}, x_{\beta,j}] = \sum_{k=0}^{i+j} (a_{\alpha,i;\beta,j}^{\alpha+\beta+b,k} x_{\alpha+\beta+b,k} + a_{\alpha,i;\beta,j}^{\alpha+\beta,k} x_{\alpha+\beta,k}) \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}, \quad (5.34)$$

where we treat $x_{\alpha+\beta+b,k} = 0$ if $b \notin \Delta$. Note that

$$(\mathcal{D} - \alpha - \beta - 2b) \left(\sum_{k=0}^{i+j} a_{\alpha,i;\beta,j}^{\alpha+\beta,k} x_{\alpha+\beta,k} \right) = \sum_{k=0}^{i+j} a_{\alpha,i;\beta,j}^{\alpha+\beta,k} (-bx_{\alpha+\beta,k} + kx_{\alpha+\beta,k-1}). \quad (5.35)$$

Hence by (5.33)-(5.35) and mathematical induction on $i + j$, we can prove that

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,i+j} = \frac{1}{b}((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha)) \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}. \quad (5.36)$$

Moreover, by (5.34) and (5.35), the coefficients of $x_{\alpha+\beta,i+j-1}$ in (5.33) shows that

$$\begin{aligned} & -ba_{\alpha,i;\beta,j}^{\alpha+\beta,i+j-1} + \frac{i+j}{b}((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha)) \\ & = \frac{i+j}{b}((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha)) + i(\lambda(\beta + b) - \varphi(\beta)) \\ & + j(\varphi(\alpha) - \lambda(\alpha + b)), \end{aligned} \quad (5.37)$$

which is equivalent to:

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,i+j-1} = \frac{1}{b}[i(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))] \quad (5.38)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$. From the coefficients of $x_{\alpha+\beta,i+j-2}$ in (5.33), we get

$$\begin{aligned} & -ba_{\alpha,i;\beta,j}^{\alpha+\beta,i+j-2} + \frac{i+j-1}{b}[i(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))] \\ & = \frac{i}{b}[(i-1)(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))] \\ & + \frac{j}{b}[i(\varphi(\beta) - \lambda(\beta + b)) + (j-1)(\lambda(\alpha + b) - \varphi(\alpha))] \\ & = \frac{i+j-1}{b}[i(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))], \end{aligned} \quad (5.39)$$

which is equivalent to:

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,i+j-2} = 0 \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}. \quad (5.40)$$

Thus by (5.33)-(5.35) and mathematical induction, we can prove that

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,k} = 0 \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}, k \leq i + j - 2. \quad (5.41)$$

Theorem 5.1. *A Lie algebra is an algebra over the simple Novikov algebra $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ (cf. (1.15)) with $b \notin \Delta$ if and only if its Lie bracket has the following form:*

$$\begin{aligned} [x_{\alpha,i}, x_{\beta,j}] &= \frac{1}{b}((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha))x_{\alpha+\beta,i+j} \\ &\quad + \frac{1}{b}[i(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))]x_{\alpha+\beta,i+j-1} \end{aligned} \quad (5.42)$$

for $\alpha, \beta \in \Delta, i, j \in \mathbb{N}$, where $\varphi : \Delta \rightarrow \mathbb{C}^+$ is a group homomorphism and $\lambda \in \mathbb{C}$ is a constant.

Proof. The necessity has been proved in the above.

To prove the sufficiency, we define operators d_1, d_2 on $\mathcal{A}_{\Delta, \mathbb{N}}$ by

$$d_1(u_{\alpha,i}) = \varphi(\alpha)u_{\alpha,i} + \lambda i u_{\alpha,i-1}, \quad d_2(u_{\alpha,i}) = \alpha u_{\alpha,i} + i u_{\alpha,i-1} \quad \text{for } \alpha \in \Delta, i \in \mathbb{N}. \quad (5.43)$$

Then d_1 and d_2 are mutually commutative derivations of $(\mathcal{A}_{\Delta, \mathbb{N}}, \cdot)$. Moreover, the algebraic operations defined in (5.42) and (1.15) have the property:

$$b[\cdot, \cdot] = [\cdot, \cdot]_{2,1} + b[\cdot, \cdot]_1, \quad \circ = \circ_b \quad (5.44)$$

in terms the notions in (3.30) and (3.31). Thus (1.15) and (5.42) define a Gel'fand-Dorfman bialgebra by Theorem 3.5. \square

Remark 5.2. (a) Up to this stage, we have not completely classified all the Lie algebras over the simple Novikov algebra $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ with $b \in \Delta$. The best informations that we have obtained are as follows. When $0 \neq b \in \Delta$,

$$a_{\alpha,i;\beta,j}^{\alpha+\beta+b,k} = \sum_{k=0}^k \binom{i}{p} \binom{j}{k-p} a_{\alpha,i-p;\beta,j+p-k}^{\alpha+\beta+b,0} \quad \text{for } \alpha, \beta \in \Delta, i, j, k \in \mathbb{N}, \quad (5.45)$$

$$a_{0,1;0,j}^{b,0} = \frac{(-1)^j(j+1)!}{6(2b)^{j-2}} a_{0,1;0,2}^{b,0}, \quad a_{0,i;0,j}^{b,0} = \frac{(-1)^{i+j}(j-i)(i+j-1)!}{6(2b)^{i+j-2}} a_{0,2;0,1}^{b,0} \quad (5.46)$$

for $1 < i, j \in \mathbb{N}$. When $b = 0$, for $\alpha, \beta \in \Delta$ and $i, j \in \mathbb{N}$,

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,k} = 0 \quad \text{for } i + j + 1 < k \in \mathbb{N}, \quad (5.47)$$

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,i+j+1} = \varphi(\alpha)\beta - \varphi(\beta)\alpha, \quad a_{\alpha,i;\beta,j}^{\alpha+\beta,i+j} = a_{\alpha,0;\beta,0}^{\alpha+\beta,0} + i(\lambda\beta - \varphi(\beta)) + j(\varphi(\alpha) - \lambda\alpha), \quad (5.48)$$

$$a_{\alpha,i;\beta,j}^{\alpha+\beta,k} = \sum_{k=0}^k \binom{i}{p} \binom{j}{k-p} a_{\alpha,i-p;\beta,j+p-k}^{\alpha+\beta,0} \quad \text{for } i+j > k \in \mathbb{N}, \quad (5.49)$$

$$[x_{0,i}, x_{0,j}] = \lambda(j-i). \quad (5.50)$$

Equation (5.50) was proved by Osborn and Zelmanov [OZ].

(b) The followings are Lie algebras over $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ with $b = 0$:

(1) For a skew-symmetric \mathbb{Z} -bilinear form $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ and a group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$, the Lie bracket is defined by:

$$[x_{\alpha,i}, x_{\beta,j}] = \phi(\alpha, \beta)x_{\alpha+\beta,i+j} + (i\varphi(\beta) - j\varphi(\alpha))x_{\alpha+\beta,i+j-1} \quad (5.51)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$. This structure is obtained by Theorem 3.8 when $[\cdot, \cdot] = [\cdot, \cdot]_{\phi, 2, 3}$ with d_i defined in (5.43).

(2) For a group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$ and a nonzero constant $\lambda \in \mathbb{C}$, the Lie bracket is defined as follows:

$$[x_{\alpha,i}, x_{\beta,j}] = (\alpha\varphi(\beta) - \beta\varphi(\alpha))x_{\alpha+\beta,i+j} + [i(\varphi(\beta) - \lambda\beta) + j(\lambda\alpha - \varphi(\alpha))]x_{\alpha+\beta,i+j-1} \quad (5.52)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$. This structure is obtained by Corollary 3.7 with d_i defined in (5.43) and $b = 0$.

(3) For a group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$, the Lie bracket is defined as follows:

$$[x_{\alpha,i}, x_{\beta,j}] = (\alpha\varphi(\beta) - \beta\varphi(\alpha) + \beta - \alpha)x_{\alpha+\beta,i+j} + (i\varphi(\beta) - j\varphi(\alpha) + j - i)x_{\alpha+\beta,i+j-1} \quad (5.53)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$. This structure is obtained by Theorem 3.6.

(c) Lie algebras over $(\mathcal{A}_{\Delta, \mathbb{N}}, \circ)$ with $0 \neq b \in \Delta$:

(1) For a group homomorphism $\varphi : \Delta \rightarrow \mathbb{C}^+$ and a constant $\lambda \in \mathbb{C}$, the Lie bracket is defined by (5.42).

(2) For group homomorphisms $\varphi, \varphi_1, \varphi_2 : \Delta \rightarrow \mathbb{C}^+$ such that $\varphi(b) = \varphi_1(b) = 0$ and a constant $\lambda \in \mathbb{C}$, the Lie bracket is defined below:

$$\begin{aligned} [x_{\alpha,i}, x_{\beta,j}] &= (\varphi_1(\alpha)\varphi_2(\beta) - \varphi_1(\beta)\varphi_2(\alpha))x_{\alpha+\beta+b,i+j} \\ &\quad + ((\alpha+b)\varphi(\beta) - (\beta+b)\varphi(\alpha))x_{\alpha+\beta,i+j} \\ &\quad + [i(\varphi(\beta) - \lambda(\beta+b)) + j(\lambda(\alpha+b) - \varphi(\alpha))]x_{\alpha+\beta,i+j-1} \end{aligned} \quad (5.54)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$. This family of Lie algebras are motivated by Theorem 4.3 and (5.42).

(d) The following Lie algebraic structure on $\mathcal{A}_{\Delta, \mathbb{N}}$ seems interesting itself, although it is not directly related to Gel'fand-Dorfman bialgebras. Assume that $0 \neq b \in \Delta$. Let $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ be a skew-symmetric \mathbb{Z} -bilinear form such that $b \notin \text{Rad}_\phi$ and let $\varphi, \varphi_1 : \Delta \rightarrow \mathbb{C}^+$ be group homomorphisms such that $\varphi(b) = 0$. We have the following Lie bracket on $\mathcal{A}_{\Delta, \mathbb{N}}$:

$$\begin{aligned} [x_\alpha, x_\beta] &= (\varphi_1(\alpha)\phi(b, \beta) - \varphi_1(\beta)\phi(b, \alpha))x_{\alpha+\beta+b} + \phi(\alpha, \beta)x_{\alpha+\beta} \\ &\quad + (i\varphi(\beta) - j\varphi(\alpha))x_{\alpha+\beta, i+j-1} \end{aligned} \quad (5.55)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$.

6 Classification III

Recall we have a commutative associative algebraic structure on $\mathcal{A}_{\Delta, \Gamma}$ defined in (1.16). As indicated in (1.18), the commutator algebras of all simple Novikov algebras $(\mathcal{A}_{\Delta, \Gamma}, \diamond_\xi)$ for $\xi \in \mathcal{A}_{\Delta, \Gamma}$ are the same. We ask whether \diamond_ξ are the all Novikov algebraic structures on $\xi \in \mathcal{A}_{\Delta, \Gamma}$ whose commutator algebras are the following Lie algebra:

$$[x_{\alpha, i}, x_{\beta, j}] = (\beta - \alpha)x_{\alpha+\beta, i+j} + (j - i)x_{\alpha+\beta, i+j-1} \quad \text{for } \alpha, \beta \in \Delta, i, j \in \mathbb{N}. \quad (6.1)$$

We shall give a confirmative answer to the case $\Gamma = \{0\}$.

Let $(\mathcal{A}_{\Delta, \Gamma}, \circ)$ be a Novikov algebra whose commutative algebra is given by (6.1). Then $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot], \circ)$ forms Gel'fand-Dorfman bialgebra, that is, (1.11) holds (cf. Theorem 2.3). Set

$$\xi = x_{0,0} \circ x_{0,0} \quad (6.2)$$

and define another algebraic operation \circ_0 on $\mathcal{A}_{\Delta, \Gamma}$ by:

$$u \circ_0 v = \xi uv \quad \text{for } u, v \in \mathcal{A}_{\Delta, \Gamma}. \quad (6.3)$$

Lemma 6.1. *The family $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot], \circ_0)$ forms a Gel'fand Dorfman bialgebra.*

Proof. It is easily seen that $(\mathcal{A}_{\Delta, \Gamma}, \circ_0)$ forms a Novikov algebra. Moreover, (1.11) holds because $\circ_0 = \diamond_\xi - \diamond_0$ and both $([\cdot, \cdot], \diamond_\xi)$ and $([\cdot, \cdot], \diamond_0)$ satisfy (1.11) (cf. (1.18) and Theorem 2.3). \square

Now we let

$$\star = \circ - \circ_0. \quad (6.4)$$

Note that the commutator algebra of $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot], \circ_0)$ is a trivial (abelian) Lie algebra. Thus the commutator algebra of the algebra $(\mathcal{A}_{\Delta, \Gamma}, \star)$ is also $(\mathcal{A}_{\Delta, \Gamma}, [\cdot, \cdot])$, and $([\cdot, \cdot], \star)$ satisfies (1.11). Specifically, we have:

$$x_{\alpha, i} \star x_{\beta, j} - x_{\beta, j} \star x_{\alpha, i} = [x_{\alpha, i}, x_{\beta, j}] = (\beta - \alpha)x_{\alpha+\beta, i+j} + (j - i)x_{\alpha+\beta, i+j-1} \quad (6.5)$$

for $\alpha, \beta \in \Delta$, $i, j \in \mathbb{N}$,

$$[w \star u, v] - [w \star v, u] + [w, u] \star v - [w, v] \star u - w \star [u, v] = 0 \quad \text{for } u, v, w \in \mathcal{A}_{\Delta, \Gamma}, \quad (6.6)$$

$$x_{0,0} \star x_{0,0} = 0. \quad (6.7)$$

Below we assume that $\Gamma = \{0\}$. We shall determine $(\mathcal{A}_{\Delta, \{0\}}, \circ)$ through $(\mathcal{A}_{\Delta, \{0\}}, \star)$.

Recall the notion in (4.1). We write

$$x_\alpha \star x_\beta = \sum_{\sigma \in \Delta} c_{\alpha, \beta}^\sigma x_\sigma \quad \text{for } \alpha, \beta \in \Delta. \quad (6.8)$$

Then by (6.3) and (6.7), we have:

$$c_{\alpha, \beta}^\sigma - c_{\beta, \alpha}^\sigma = \delta_{\sigma, \alpha+\beta}(\beta - \alpha) \quad \text{for } \alpha, \beta, \sigma \in \Delta, \quad (6.9)$$

$$c_{0,0}^\sigma = 0 \quad \text{for } \sigma \in \Delta. \quad (6.10)$$

Moreover, for $\alpha, \beta, \gamma \in \Delta$, (6.6) gives

$$\begin{aligned} & [x_\gamma \star x_\alpha, x_\beta] - [x_\gamma \star x_\beta, x_\alpha] + [x_\gamma, x_\alpha] \star x_\beta - [x_\gamma, x_\beta] \star x_\alpha - x_\gamma \star [x_\alpha, x_\beta] \\ &= \sum_{\sigma \in \Delta} [(\beta - \sigma)c_{\gamma, \alpha}^\sigma x_{\sigma+\beta} - (\alpha - \sigma)c_{\gamma, \beta}^\sigma x_{\sigma+\alpha} + ((\alpha - \gamma)c_{\gamma+\alpha, \beta}^\sigma \\ & \quad - (\beta - \gamma)c_{\gamma+\beta, \alpha}^\sigma - (\beta - \alpha)c_{\gamma, \alpha+\beta}^\sigma)x_\sigma] \\ &= \sum_{\sigma \in \Delta} [(2\beta - \sigma)c_{\gamma, \alpha}^{\sigma-\beta} - (2\alpha - \sigma)c_{\gamma, \beta}^{\sigma-\alpha} + (\alpha - \gamma)c_{\gamma+\alpha, \beta}^\sigma \\ & \quad - (\beta - \gamma)c_{\gamma+\beta, \alpha}^\sigma - (\beta - \alpha)c_{\gamma, \alpha+\beta}^\sigma]x_\sigma = 0, \end{aligned} \quad (6.11)$$

which is equivalent to:

$$(2\beta - \sigma)c_{\gamma, \alpha}^{\sigma-\beta} - (2\alpha - \sigma)c_{\gamma, \beta}^{\sigma-\alpha} + (\alpha - \gamma)c_{\gamma+\alpha, \beta}^\sigma - (\beta - \gamma)c_{\gamma+\beta, \alpha}^\sigma - (\beta - \alpha)c_{\gamma, \alpha+\beta}^\sigma = 0 \quad (6.12)$$

for $\alpha, \beta, \gamma, \sigma \in \Delta$.

Letting $\alpha = \gamma = 0$ and $\beta \neq 0$ in (6.12), we get:

$$\sigma c_{0, \beta}^\sigma - \beta c_{\beta, 0}^\sigma - \beta c_{0, \beta}^\sigma = 0 \quad (6.13)$$

by (6.10), which implies

$$(\sigma - 2\beta)c_{0, \beta}^\sigma = -\beta^2 \delta_{\sigma, \beta} \quad (6.14)$$

by (6.9). Hence

$$c_{0,\beta}^\sigma = 0 \quad \text{for } \beta, \sigma \in \Delta, \sigma \neq \beta, 2\beta, \quad (6.15)$$

$$c_{0,\beta}^\beta = \beta \quad \text{for } \beta \in \Delta. \quad (6.16)$$

Next we let $\gamma = 0$, $\sigma = 2(\alpha + \beta)$, $\alpha \neq \pm\beta$ and $\alpha\beta \neq 0$ in (6.12), and obtain:

$$\alpha c_{\alpha,\beta}^{2(\alpha+\beta)} - \beta c_{\beta,\alpha}^{2(\alpha+\beta)} - (\beta - \alpha) c_{0,\alpha+\beta}^{2(\alpha+\beta)} = 0, \quad (6.17)$$

by (6.15). Moreover, by (6.9), (6.17) can be written as:

$$(\alpha - \beta) c_{\alpha,\beta}^{2(\alpha+\beta)} - (\beta - \alpha) c_{0,\alpha+\beta}^{2(\alpha+\beta)} = 0, \quad (6.18)$$

which implies:

$$c_{\alpha,\beta}^{2(\alpha+\beta)} = -c_{0,\alpha+\beta}^{2(\alpha+\beta)} \quad \text{for } 0 \neq \alpha, \beta \in \Delta, \alpha \neq \pm\beta. \quad (6.19)$$

When $\alpha = 0$, $\sigma = 2(\beta + \gamma)$ and $\beta\gamma \neq 0$, $\beta \neq \pm\gamma$ in (6.12), we have:

$$2(\beta + \gamma) c_{\gamma,\beta}^{2(\beta+\gamma)} - \gamma c_{\gamma,\beta}^{2(\beta+\gamma)} - (\beta - \gamma) c_{\gamma+\beta,0}^{2(\beta+\gamma)} - \beta c_{\gamma,\beta}^{2(\beta+\gamma)} = 0 \quad (6.20)$$

by (6.15). Moreover, by (6.9) and (6.19), (6.20) implies:

$$-2\beta c_{0,\beta+\gamma}^{2(\beta+\gamma)} = 0. \quad (6.21)$$

Thus

$$c_{\gamma,\beta}^{2(\beta+\gamma)} = c_{0,\beta+\gamma}^{2(\beta+\gamma)} = 0. \quad (6.22)$$

For any $0 \neq \tau \in \Delta$, we let $\beta = 2\tau$ and $\gamma = -\tau$. Then β and γ satisfy our assumption. So

$$c_{0,\tau}^{2\tau} = c_{0,2\tau+(-\tau)}^{2(2\tau+(-\tau))} = 0 \quad \text{for } \tau \in \Delta. \quad (6.23)$$

Assuming that $\alpha = 0$ and $\sigma \neq \beta + \gamma$ in (6.12), we get:

$$\sigma c_{\gamma,\beta}^\sigma - \gamma c_{\gamma,\beta}^\sigma - \beta c_{\gamma,\beta}^\sigma = 0 \quad (6.24)$$

by (6.15) and (6.22)-(6.23). Thus

$$c_{\gamma,\beta}^\sigma = 0 \quad \text{for } \beta, \gamma, \sigma \in \Delta, \sigma \neq \beta + \gamma. \quad (6.25)$$

Supposing that $\gamma = 0$, $\alpha \neq \beta$ and $\sigma = \alpha + \beta$ in (6.12), we get:

$$(\beta - \alpha) c_{0,\alpha}^\alpha + (\beta - \alpha) c_{0,\beta}^\beta + \alpha c_{\alpha,\beta}^{\alpha+\beta} - \beta c_{\beta,\alpha}^{\alpha+\beta} - (\beta - \alpha) c_{0,\alpha+\beta}^{\alpha+\beta} = 0, \quad (6.26)$$

which is equivalent to:

$$(\alpha - \beta) c_{\alpha,\beta}^{\alpha+\beta} = \beta(\alpha - \beta). \quad (6.27)$$

So we have:

$$c_{\alpha,\beta}^{\alpha+\beta} = \beta \quad \text{for } \alpha, \beta \in \Delta, \alpha \neq \beta. \quad (6.28)$$

Letting $\alpha = 3\beta$, $\gamma = \beta \neq 0$ and $\sigma = 5\beta$ in (6.12), we get:

$$-3\beta c_{\beta,3\beta}^{4\beta} - \beta c_{\beta,\beta}^{2\beta} + 2\beta c_{4\beta,\beta}^{5\beta} + 2\beta c_{\beta,4\beta}^{5\beta} = 0, \quad (6.29)$$

which is equivalent to:

$$-9\beta^2 - \beta c_{\beta,\beta}^{2\beta} + 2\beta^2 + 8\beta^2 = 0 \quad (6.30)$$

by (6.28). Thus we have:

$$c_{\beta,\beta}^{2\beta} = \beta \quad \text{for } \beta \in \Delta. \quad (6.31)$$

By (6.25), (6.28) and (6.31), we have:

$$x_\alpha \star x_\beta = \beta x_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (6.32)$$

Hence by (6.2)-(6.4), Lemma 6.1 and (6.32), we obtain the main result in this section:

Theorem 6.2. The set $\{(\mathcal{A}_{\Delta,\{0\}}, \diamond_\xi) \mid \xi \in \mathcal{A}_{\Delta,\{0\}}\}$ enumerates all the Novikov algebraic structures over $\mathcal{A}_{\Delta,\{0\}}$ whose commutator algebras are given by (6.1).

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